Parametric Analysis of Stability Conditions for a Satellite with a Gravitation Stabilizer

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Abstract. A software system elaborated on the basis of the computer algebra software package "Mathematica" has been employed in investigations of stability of a relative equilibrium position for an uncontrolled satellite having a gravitation stabilizer on a circular orbit. Domains of various degrees of Poincaré instability have been revealed in the space of introduced parameters. Under the assumption that the potential system is unstable (the degree of instability being even) the problem of possibility of its gyroscopic stabilization is considered. Parametric analysis of stability conditions has been conducted and two-parameter sections of the system domains of gyroscopic stabilization have been constructed with the aid of an applied program intended for graphic representation of the solution of the system of algebraic inequalities.[1]

1 Introduction

The investigation of problems of both stability and stabilization of nonlinear or linearized models of mechanical systems often leads to the problem of "parametric analysis" of the conditions (inequalities) obtained. In the case of parametric analysis it is important to have the possibility to estimate the domain of values of the parameters under which a desired system state is provided. It is naturally hard to hope to obtain any readable analytical results for the models which have high dimensions and contain many parameters. At this stage one can efficiently use the software packages of computer algebra (SPCA) as well as the corresponding software developed on the basis of these software packages.

The author has developed the software PASI [1] intended for obtaining the solutions of the systems of multiparametric algebraic inequalities. The software PASI represents a package of interactive user programs which are executed in interpretation mode in the environment of SPCA "Mathematica". If the system of inequalities under scrutiny has more than two parameters then employing the above-mentioned software the user can construct some sections (cuts) of the space of all the parameters by the planes of the two parameters x_1, x_2 chosen by him (the rest of the parameters being fixed). As a result, an initial system of algebraic inequalities of several variables is reduced after the corresponding automatic transformations to a system of one-parameter polynomials with numerical coefficients

$$F_i(x_j) = a_n^{(i)} x_j^n + a_{n-1}^{(i)} x_j^{n-1} + \ldots + a_1^{(i)} x_j + a_0^i \geq 0 \ \ (\leq 0), \qquad (i = 1, \ldots, k_1;\ j = 1, 2) \qquad (1)$$

and to a system of two-parameter inequalities with numerical coefficients of the form

$$F_i(x_1, x_2) \geq 0 \ \ (\leq 0), \quad (i = k_1 + 1, \ldots, k), \qquad (2)$$

which are non-resolvable explicitly with respect to one of the parameters. To the end of solving the system of inequalities (1) we have developed an applied program **SolveSystemInequality**, which searches for the solution in symbolic-numeric form as a union of non-overlapping intervals of the numerical axis. For the purpose of solving the system of inequalities (2) with account of solutions of inequalities (1) an applied program **Region** has been developed which finds and performs a graphical construction of a 2D domain in the plane of parameters x_1 and x_2. A list of symbolic-numeric inequalities, names (identifiers) of parameters and intervals of variation of their values with respect to abscissa and ordinate axes are assigned as the input for this applied program. The algorithms for graphic representation of solutions of the systems of algebraic inequalities (1) and (2) in its complete form, examples of loading applied programs, their usage as well as specific traits of the programs **SolveSystemInequality**, **Region** are described in detail in [2]. These two programs form a kernel of the software PASI.

[1] The work has been partially supported by the Russian Foundation for Basic Research (grant No. 00-01-00730).

Note that in the last version of SPCA "Mathematica" there appeared the user package **Inequality-Graphics** (its author is Roger Germundsson, 2000), which is also intended for constructing the graphic 2D and 3D images of the solution of the system of algebraic inequalities. Maple V is capable of finding the solutions only for the systems of linear inequalities.

It is known that the stability of rotation of conservative satellite systems of rigid bodies as well as secular stability can be bound up also with gyroscopic stabilization [3], [4]. The objective of this work was to employ the developed software, conduct a parametric analysis of the obtained stability conditions, and construct the domains of gyroscopic stabilization for a well-known model [5].

2 Problem Statement

Consider a problem of the stability of a position of relative equilibrium in the orbital coordinate system for an uncontrolled satellite with a gravitational stabilizer. The stabilizer represents a rigid rod having the point mass at its free end. The rod is connected to the satellite with the aid of a 2-degree-of-freedom suspension. The rotation axes of the rod coincide with the direction of the axes of pitch and bank. The hinges are supposed to be frictionless. The system mass center performs some passive motion along the Kepler circular orbit. The system is influenced by some gravitation moment (other moments being negligibly small). In undisturbed motion the system principal central axes of inertia coincide with orbital axes, and the rod is oriented along the radius of the orbit.

In previous papers (see, e.g., the survey [6]) for a similar type model the authors obtained the conditions of asymptotic stability under various laws of control of the stabilizer, determined optimal system's parameters and investigated the influence of different effects on the system dynamics.

The linearized equations of motion are decomposed into two subsystems [5]: in the pitch channel (θ) and in the yaw-and-bank channel (ψ , φ), respectively.

$$M_1\,\ddot q_1 \;+\; K_1\,q_1 \;=\; 0\,; \qquad M_2\,\ddot q_2 \;+\; G\,\dot q_2 \;+\; K_2\,q_2 \;=\; 0 \tag{3}$$

$$K_1 = 3\omega^2\begin{pmatrix} b-a & f \\ f & f\end{pmatrix}; \qquad M_1 = \begin{pmatrix} c & f \\ f & d\end{pmatrix} > 0\,; \qquad q_1 = \begin{pmatrix}\theta \\ \delta_1\end{pmatrix}; \qquad q_2 = \begin{pmatrix}\psi \\ \varphi \\ \delta_2\end{pmatrix};$$

$$K_2 = \omega^2\begin{pmatrix} c-b & 0 & 0 \\ 0 & 4(c-a) & 4f \\ 0 & 4f & 3f+d\end{pmatrix}; \qquad M_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & f \\ 0 & f & d\end{pmatrix} > 0\,; \qquad G = \omega\begin{pmatrix} 0 & \mathrm{g} & 0 \\ -\mathrm{g} & 0 & 0 \\ 0 & 0 & 0\end{pmatrix},$$

where ω is the modulus of the orbital angular velocity; δ_1 , δ_2 are rotation angles of the rod with respect to the satellite body;

$$a = J_y\,; \quad b = J_x + m\,r\,(\,r+l\,) + \frac{1}{3}\,m\,l^2 + m_0\,(\,r+l\,)^2\,; \quad c = b + J_z - J_x\,;$$

$$d = (\,\frac{1}{3}\,m + m_0\,)\,l^2\,; \quad f = (\,\frac{1}{2}\,m + m_0\,)\,r\,l \,+\, d\,; \quad \mathrm{g} = J_z - J_x - J_y\,;$$

m , m_0 are the masses of the rod and the point load at the end, respectively; $l > 0$ is the rod length; $r > 0$ is the distance from the mass center of the system to the point of attachment of the rod; J_x , J_y , and J_z are principal inertia moments of the satellite.

Equations (3) may be interpreted as the equations of oscillations of a mechanical system influenced by potential forces (with the matrices K_1 , K_2) and gyroscopic forces (with the matrices G). These forces are determined by the gravitation forces as well as by orbital motion. M_1 and M_2 play the role of diagonal blocks of the kinetic energy matrix.

In [7], on the basis of Lyapunov's second method, the bundle of first integrals for equations (3) with indeterminate multipliers was written. Under a specific numerical distribution of masses in the mechanical system a domain for selecting these multipliers from Sylvester's conditions of positive definiteness of the bundle of integrals was constructed. Unlike [7], below we will show that the question of gyroscopic stabilization of the system can be solved in analytical form on the basis of Kelvin-Chetayev's theorems [3] of the first approximation.

Introduce four dimensionless parameters:

$$\alpha = \frac{c - b}{a} = \frac{J_z - J_x}{J_y} ; \quad \gamma = \frac{b - a}{c} ; \quad p_1 = \frac{d}{f} ; \quad p_2 = \frac{f}{c} . \tag{4}$$

The physically obtainable values of the parameters lie within the intervals:

$$|\alpha| < 1 ; \quad |\gamma| < 1 ; \quad 0 < p_1 < 1 ; \quad 0 < p_2 < 1 . \tag{5}$$

The secular equation of the system (3) : $\Delta(\lambda^2) = \Delta_1(\lambda^2) * \Delta_2(\lambda^2) = 0$ contains λ only in even powers. Here the characteristic determinants for the pitch channel and the yaw-and-bank channel are, respectively :

$$\Delta_1 = \begin{vmatrix} \lambda^2 + 3\omega^2\gamma & p_2(\lambda^2 + 3\omega^2) \\ \lambda^2 + 3\omega^2 & \lambda^2 p_1 + 3\omega^2 \end{vmatrix} = \lambda^4 (p_1 - p_2) + 3\omega^2\lambda^2 (1 + \gamma p_1 - 2 p_2) + 9\omega^4 (\gamma - p_2) ;$$

$$\Delta_2 = \begin{vmatrix} \lambda^2 + \omega^2\alpha & \lambda\omega(\alpha - 1) & 0 \\ \lambda\omega(\gamma - 1)(\alpha - 1) & \lambda^2(1 + \gamma\alpha) + 4\omega^2(\alpha + \gamma) & p_2(\alpha + 1)(\lambda^2 + 4\omega^2) \\ 0 & \lambda^2 + 4\omega^2 & \lambda^2 p_1 + (3 + p_1)\omega^2 \end{vmatrix} =$$

$$= v_6 \lambda^6 + v_4 \lambda^4 + v_2 \lambda^2 + v_0 = \lambda^6 (p_1 - p_2 + \alpha (\gamma p_1 - p_2))$$
$$+ \lambda^4 (3 + 3\alpha\gamma + 2 p_1 + 3\alpha p_1 + \alpha^2 p_1 + 3\gamma p_1 + 3\alpha\gamma p_1 - 8 p_2 - 9\alpha p_2 - \alpha^2 p_2)\omega^2$$
$$+ \lambda^2 (3 + 9\alpha + 3\alpha^2 + 9\gamma + 6\alpha\gamma + p_1 + 3\alpha p_1 + 5\alpha^2 p_1 + 3\gamma p_1 + 6\alpha\gamma p_1 - 16 p_2 - 24\alpha p_2 - 8\alpha^2 p_2)\omega^4$$
$$+ 4\omega^6 \alpha ((\gamma + \alpha)(3 + p_1) - 4 p_2 (\alpha + 1)) .$$

It is known that the stability in such systems is possible only in the case when all the roots of the polynomial Δ with respect to λ^2 are negative and real. The corresponding criterion [8] of negativity and reality of the roots for the $n-$ th degree polynomial has been algorithmically implemented by the author (see [9]) with the aid of SPCA "Mathematica". The algebraic conditions, which provide for the desired properties of the roots, respectively, for the pitch and yaw-and-bank channels,

$$\begin{cases} p_1 - p_2 > 0 , \quad 1 + \gamma p_1 - 2 p_2 > 0 , \quad \gamma - p_2 > 0 , \\ p_1^2 \gamma^2 + (4 p_2 (1 - p_1) - 2 p_1)\gamma + (1 - 4 p_2 (1 - p_1)) > 0 \end{cases} \tag{6}$$

$$\begin{cases} v_6 > 0 , \quad v_4 > 0 , \quad v_2 > 0 , \quad v_0 > 0 , \\ Dis = v_4^2 v_2^2 - 4 v_2^3 v_6 - 4 v_4^3 v_0 + 18 v_6 v_4 v_2 v_0 - 27 v_0^2 v_6^2 > 0 \end{cases} \tag{7}$$

will define the relations between the system parameters, the satisfaction of which makes stable the motion of a satellite with a stabilizer. The latter inequality in (7) has also been obtained in analytical form with the aid of a special program and written down in the form of an explicit dependence on the parameters (4). But it is too awkward to be given here. To emphasize the complexity of the analyzed expression it is sufficient to note only that the maximum powers of the parameters α, γ, p_1, p_2 in Dis are 8, 4, 4, 3, respectively.

3 Stability in the Pitch Channel

Note that the first of the conditions (6) is satisfied due to definite positiveness of the matrix of kinetic energy M_1. It can easily be shown that the last inequality in (6) also holds. To this end let us write down the discriminant of this polynomial with respect to γ : $16 (p_1 - 1)^2 p_2 (p_2 - p_1)$, which is always negative for $p_1 > p_2$. Due to both above assumptions and positivity of the coefficient for γ^2 it is possible to conclude that the polynomial under scrutiny assumes only positive values. Graphically it is easy to show (see Fig. 1) that for two inequalities in (6), which have not yet been considered, the satisfaction of the condition $1 + \gamma p_1 - 2 p_2 > 0$ is provided by the inequality $\gamma - p_2 > 0$.

So, the solution in the pitch channel is stable if and only if $\gamma - p_2 > 0$.

Before proceeding to the parametric analysis of stability conditions in the yaw-and-bank channel, let us define the domain of real values of parameters α and γ with regard for conditions (5) and $\gamma - p_2 > 0$. Denote $k = a/b$ and after several simple transformations the following relation can be obtained :

$$\gamma = \frac{1 - k}{1 + k\alpha} , \quad \text{where} \quad 0 < k < 1 - \frac{f}{b} . \tag{8}$$

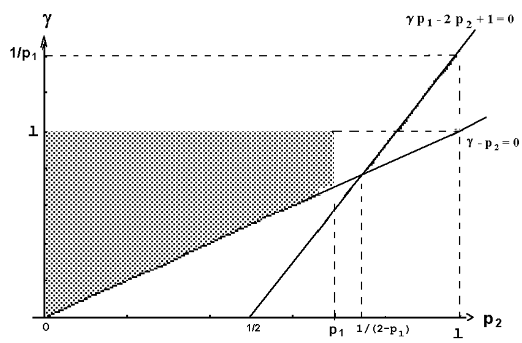

Fig. 1. Domain of stability in the pitch channel

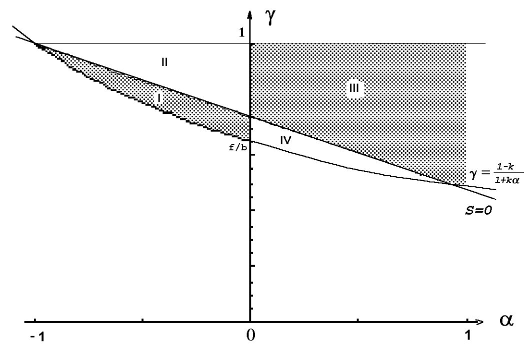

Fig. 2. Domains having different degrees of instability

4 Gyroscopic stabilization

Kelvin-Chetayev's theorems [3] allow one to start the investigation of stability of the trivial solution for the yaw-and-bank channel from the analysis of the matrix of potential forces (in notation of (4)) :

$$K_2 \, = \, 4\,\omega^2 \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha+\gamma & 4\,p_2\,(\alpha+1) \\ 0 & 1 & 3+p_1 \end{pmatrix} .$$

Even (odd) character of the degree of Poincaré instability is determined by positivity (negativity) of the determinant of the matrix K_2. Note that $\det K_2 = v_0$.

The relations $\alpha = 0$ and $S \, = \, (\,\gamma + \alpha\,)\,(\,3 + p_1\,) \, - \, 4\,p_2\,(\,\alpha + 1\,) \, = \, 0$ define in the space of parameters the surfaces which separate the domains having different degrees of instability. For example, Fig. 2 qualitatively shows these domains for the fixed values of parameters p_1 and p_2. Here, respectively, the instability domains have : I – an even degree ; II, IV – odd degree ; III – zero degree. The shaded area is bounded by the lines obtained from (8) by substituting the ultimate possible values of $k = 0$ and $k = 1 - f/b$. In domain I the subsystem of yaw-and-bank is potentially unstable, but gyroscopic stabilization is possible in it. For domain III, the addition of gyroscopic forces to potential ones helps to retain the property of stability of the motion under scrutiny. Gyroscopic stabilization is impossible in domains II and IV.

Now consider the question of possibility of gyroscopic stabilization of the system when the condition $J_y > J_x > J_z$ for the satellite inertia moments holds. Note that in this case $\alpha < 0$, and hence, the subsystem of yaw-and-bank is potentially unstable (i.e. K_2 is not a positive definite matrix). The domain of gyroscopic stabilization, which has an even degree of instability, can obviously be defined as a solution of the system of inequalities:

$$\alpha \, < \, 0\,; \qquad (\,\gamma + \alpha\,)\,(\,3 + p_1\,) - 4\,p_2\,(\,\alpha + 1\,) \, < \, 0\,. \tag{9}$$

For the purpose of finding the property of gyroscopic stabilization it is necessary to determine in what part of the domain (9) the inequalities (7) are valid. Note that $v_6 \, = \, \det M_2$, hence, the first of the conditions (7) is satisfied due to definite positiveness of the matrix of kinetic energy M_2.

Parametric analysis of stability conditions (7) has been conducted with the aid of the applied program **Region** intended for a graphic representation of the solution of the system of algebraic symbolic-numeric inequalities. Within the framework of this analysis the initial 4-dimensional space of parameters was partitioned by the planes of two chosen parameters (other parameters being fixed). As a result, several 2-parameter sections (cuts), which are of interest for us, have been constructed.

For example, let us fix the parameters of the stabilizer by the values of : $p_1 = 0.8$, $p_2 = 0.7$. In this case, the expressions from (7) will write:

$$
\begin{aligned}
v_0 &= 4\alpha\,(-2.8 + \alpha + 3.8\,\gamma)\,;\\
v_2 &= -7.4 + 1.4\,\alpha^2 + 11.4\,\gamma + \alpha\,(-5.4 + 10.8\,\gamma)\,;\\
v_4 &= -1. + 0.1\,\alpha^2 + 2.4\,\gamma + \alpha\,(-3.9 + 5.4\,\gamma)\,;
\end{aligned}
$$

$$
\begin{aligned}
Dis = {}& 0.0004\,\alpha^8 + \alpha^7\,(0.096 - 0.106\,\gamma) - 1.14\,\alpha^6\,(-1.622 + \gamma)\,(-0.904 + \gamma)\\
& - 51.341\,\alpha^5\,(-1.221 + \gamma)\,(-1.057 + \gamma)\,(-0.889 + \gamma)\\
& - 158.995\,\alpha^4\,(-1.683 + \gamma)\,(-1.275 + \gamma)\,(-0.916 + \gamma)\,(-0.787 + \gamma)\\
& + 424.57\,\alpha^3\,(-1.311 + \gamma)\,(-1.031 + \gamma)\,(0.572 - 1.496\,\gamma + \gamma^2)\\
& - 257.227\,\alpha^2\,(-1.214 + \gamma)\,(-0.765 + \gamma)\,(0.465 - 1.298\,\gamma + \gamma^2)\\
& - 88.282\,\alpha\,(-1.827 + \gamma)\,(-0.573 + \gamma)\,(0.561 - 1.493\,\gamma + \gamma^2)\\
& + 83.174\,(0.686 - 1.625\,\gamma + \gamma^2)\,(0.422 - 1.298\,\gamma + \gamma^2)\,.
\end{aligned}
$$

The result of operation of the applied program

$$\mathbf{Region}[\,\{\,v_4 > 0\,,\ v_2 > 0\,,\ v_0 > 0\,,\ Dis > 0\,\}\,,\,\{\,\alpha\,,\,-1\,,\,0\,\}\,,\,\{\,\gamma\,,\,0.7\,,\,1\,\}\,]$$

is shown in Fig. 3.

As it is obvious from Fig. 3, the gyroscopic stabilization is possible not for the whole domain I (Fig. 2) but only for its shaded "needle-shape" part.

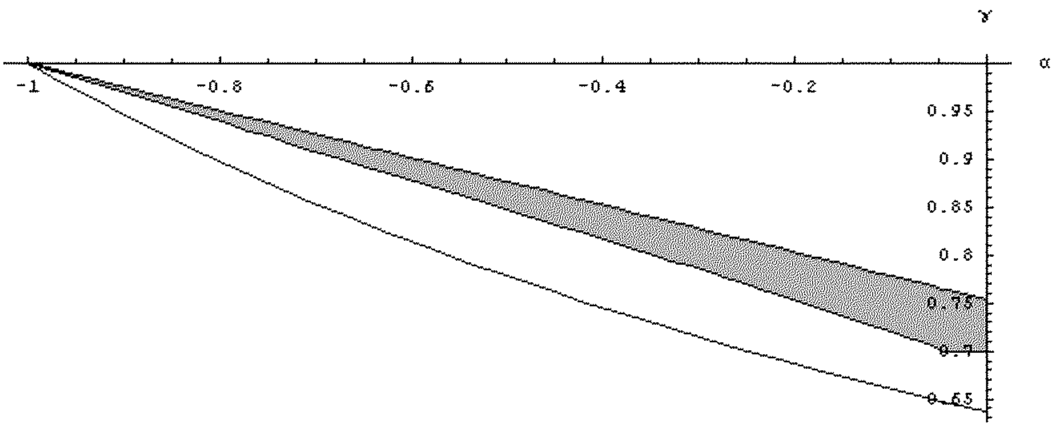

Fig. 3. Domain of gyroscopic stabilization

5 Conclusion

It is necessary to emphasize that the problems of reliability and precision of computations as well as the problems of explicitness and speeding-up of the process of investigations can be partially solved when SPCA is chosen in the capacity of the software tool. Side by side with employment of the SPCA (in the capacity of a "calculator") for solving a definite problem the approach, which implies the development of some software for solving a definite class of problems on the basis of the internal programming language of the SPCA (in our case "Mathematica"), is quite important. Practically, the overall above analysis has been conducted with the use of this software.

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