On the Properties of Families of First Integrals

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Abstract. The paper discusses some problems of investigation of peculiarities of families of invariant manifolds, which give a stationary value to the families of first integrals of differential equations of motion of mechanical systems. The method of investigation is based on using the envelope of such families of first integrals. Examples illustrating application of the method are given. The computer algebra system MATHEMATICA was applied in computations.

One of the problems arising in the process of investigation of the phase space of systems of differential equations, which assumes several first integrals, is finding out and analysis of qualitative properties of peculiar sets of both these integrals and their families, in particular, the properties of invariant sets on which some elements of the first integrals algebra reach their stationary values. Such sets will further be called the invariant manifolds of steady motions (IMSMs). The latter statement is valid due to the statements made in the spirit of the Routh-Lyapunov theorem asserting that – in the case of sufficient smoothness and under some requirements of nonsingularity imposed on the equations of stationarity of first integrals – their solutions are indeed the invariants sets (and, as a rule, are the manifolds) of the initial system of differential equations. As far as co-dimension of one is concerned, the analysis of peculiarities in first-integral algebra itself is also of substantial interest. Enveloping first integrals of such families represent one of the simple forms of peculiarities for the families of first integrals.

Since not only one first integral can reach its stationary value on a given IMSM or on their family, for the purpose of obtaining complete information on the IMSM's properties it is desirable to know the complete collection of such first integrals. This is especially useful in the case of algebraic first integrals. Here we deal with some analogy of a construction with a field of rational functions on a surface (or on a curved line) in algebraic geometry. Constructing of all first integrals, which reach their stationary values on some family of invariant manifolds, represents a special serious problem even for the low-dimensional problems of mechanics.

In the present paper we restrict ourselves to the problem of the construction and use of envelopes for the families of first integrals.

Note only that other techniques (useful for the analysis of the families of IMSMs), which are needed to the end of obtaining "additional" first integrals for their one-parameter families, are also possible. For example, let there be a one-parameter family of first integrals (with the parameter λ) for a system of differential equations:

$$K = V_1 + \lambda V_2 + \lambda^2 V_3 = c_1 + \lambda c_2 + \lambda^2 c_3.$$

(Here c_1, c_2, c_3 are constants of integrals V_1, V_2, V_3). Then it is possible to construct, generally speaking, two families of first integrals by solving the following equation with respect to λ.

$$(V_1 - c_1) + \lambda(V_2 - c_2) + \lambda^2(V_3 - c_3) = 0.$$

As a result, we have

$$\lambda_{1,2} = \frac{1}{2}[-(V_2 - c_2) \pm \sqrt{(V_2 - c_2)^2 - 4(V_1 - c_1)(V_3 - c_3)}](V_3 - c_3)^{-1}.$$

Often such "dual" first integrals allow to obtain some additional information on IMSMs corresponding to the initial family K in terms of arbitrary constants c_1, c_2, c_3 of these integrals.

Let us turn back to our principal theme and give specific examples of the construction and application of envelopes for the first integrals in problems of mechanics. Let us start with the case of quadratic first integrals.

The case of the Lagrange top in the force field of constant gravity As is well known, the differential equations of motion for the Lagrange top [1]:

$$A\dot{p} = (A-C)qr_0 + z_0\gamma_2, \quad \dot{\gamma_1} = r_0\gamma_2 - q\gamma_3,$$
$$A\dot{q} = (C-A)r_0 p - z_0\gamma_1, \quad \dot{\gamma_2} = p\gamma_3 - r_0\gamma_1,$$
$$C\dot{r} = 0, \qquad \dot{\gamma_3} = q\gamma_1 - p\gamma_2$$

have a one-parameter family of first integrals

$$K = \frac{1}{2}(Ap^2 + Aq^2 + 2z_0\gamma_3) - \lambda(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3) + \frac{1}{2}A\lambda^2({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2). \tag{1}$$

As it is obvious from the following steady-state equations, the stationary value to the elements of this family

$$\begin{aligned} &\frac{\partial K}{\partial p} = A(p-\lambda\gamma_1) = 0, && \frac{\partial K}{\partial q} = A(q-\lambda\gamma_2) = 0, \\ &\frac{\partial K}{\partial \gamma_1} = -\lambda A(p-\lambda\gamma_1) = 0, && \frac{\partial K}{\partial \gamma_2} = -\lambda A(q-\lambda\gamma_2) = 0, \\ &\frac{\partial K}{\partial \gamma_3} = z_0 - \lambda C r_0 + \lambda^2 A\gamma_3 = 0 \end{aligned} \tag{2}$$

is given by the elements of the family of invariant manifolds of regular precessions

$$p - \lambda\gamma_1 = 0, \quad q - \lambda\gamma_2 = 0, \quad z_0 - \lambda C r_0 + \lambda^2 A\gamma_3 = 0. \tag{3}$$

The first integral, which serves as the envelope for the family of the integrals (1), can easily be found here by the standard algorithm, i.e. by computing the derivative of K with respect to λ and equating it to zero:

$$\frac{\partial K}{\partial \lambda} = -(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3) + \lambda A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) = 0.$$

From the resultant equation we obtain:

$$\lambda = \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}. \tag{4}$$

After that, by substituting this value of λ into expression (1), as a result of elementary transformations we have the formula for the first integral which envelopes the family (1):

$$2\Lambda = (Ap^2 + Aq^2 + 2z_0\gamma_3) - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)^2}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)} = const.$$

It represents a rational function of first integrals, which enters the initial family K. The conditions of stationarity for the integral Λ are as follows:

$$\begin{aligned} &\frac{\partial \Lambda}{\partial p} = A[p - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}\gamma_1] = 0, \\ &\frac{\partial \Lambda}{\partial q} = A[q - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}\gamma_2] = 0, \\ &\frac{\partial \Lambda}{\partial \gamma_1} = -\frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}A[p - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}\gamma_1] = 0, \\ &\frac{\partial \Lambda}{\partial \gamma_2} = -\frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}A[q - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}\gamma_2] = 0, \\ &\frac{\partial \Lambda}{\partial \gamma_3} = z_0 - \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}Cr_0 + \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)^2}{A^2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)^2}A\gamma_3 = 0. \end{aligned} \tag{5}$$

Direct comparison of the latter equations with the conditions of stationarity (2) of the family K shows that they formally coincide when the expression

$$\frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}$$

in (5) is replaced with λ.

When substituting the values of constants of the first integrals of squares and directional cosines

$$Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3 = m, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

into the formula (4), we obtain that the stationary value for the enveloping first integral is given only by such invariant manifolds (3) for which the restriction of the form

$$\lambda = \frac{m}{A}.$$

is imposed on λ.

To the end of defining which part of IMSM (3) is represented by the solutions for the system (5), let us use equations (5) in order to remove $p,\ q$ from the square integral. As a result, we have:

$$Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3 = m(\gamma_1^2 + \gamma_2^2) + Cr_0\gamma_3 = m - m\gamma_3^2 + Cr_0\gamma_3 = m,$$

or

$$\gamma_3(-m\gamma_3 + Cr_0) = 0.$$

Since from the latter equation (5) it follows that

$$\gamma_3 = \frac{mCr_0 - z_0A}{m^2},$$

then for $(-m\gamma_3 + Cr_0)$ the expression z_0Am^{-1} is obtained, which can vanish only for $z_0 = 0$ or $m \to \infty$. Consequently, for the precessions corresponding to the enveloping integral, $\gamma_3 = 0$, and the precession angular rate is $\lambda = z_0(Cr_0)^{-1}$.

So, the enveloping integral reaches its stationary value on a special subset of the family of invariant manifolds, which correspond to the initial set of first integrals K.

If $p,\ q$, and γ_3 are removed from the integral of squares with the aid of equations (3), then for λ we obtain the fourth-degree equation:

$$A^2\lambda^4 - Am\lambda^3 + Cr_0z_0\lambda - z_0^2 = 0. \tag{6}$$

Consequently, on the hypersurface of the level of the integral of squares for a fixed value of the constant m there may be up to four regular precessions, for which the values of angular rates of the precessions are real roots of the latter equation.

If equation (6) is rewritten in the form

$$A\lambda^3(A\lambda - m) + z_0(Cr_0\lambda - z_0) = 0,$$

then for $A\lambda - m = 0$ this equation implies that the restriction $\lambda = z_0(Cr_0)^{-1}$ is imposed into the angular rate of the precession, and m in this case turns out to be equal to $z_0A(Cr_0)^{-1}$.

The substitution of the latter value of m into equation (6) allows one to factorize the latter:

$$Cr_0(A^2\lambda^4 - \frac{z_0A^2}{Cr_0}\lambda^3 + Cr_0z_0\lambda - z_0) = (\lambda - \frac{z_0}{Cr_0})(A^2\lambda^3 + Cr_0z_0)Cr_0 = 0.$$

Hence, it follows that besides the precession with $\lambda = z_0(Cr_0)^{-1}$, $\gamma_3 = 0$, on the hypersurface of the square integral having the value of the constant $m = z_0A(Cr_0)^{-1}$ there is another precession with the angular rate of $\lambda = \sqrt[3]{(-Cr_0z_0)A^{-2}}$.

The case of Lagrange top in the central force field In the case of Lagrange top in the central force field [2] the set of special regular precessions, which are separated by the enveloping first integral, is substantially richer. The differential equations of motion of a body in this case are as follows:

$$\begin{aligned} A\dot{p} &= (A - C)qr_0 + z_0\gamma_2 - \mu(A - C)\gamma_2\gamma_3, & \dot{\gamma_1} &= r_0\gamma_2 - q\gamma_3, \\ A\dot{q} &= (C - A)r_0p - z_0\gamma_1 - \mu(C - A)\gamma_3\gamma_1, & \dot{\gamma_2} &= p\gamma_3 - r_0\gamma_1, \\ C\dot{r} &= 0, & \dot{\gamma_3} &= q\gamma_1 - p\gamma_2. \end{aligned} \tag{7}$$

For these equations there is a one-parameter family of first integrals (7) similar to the family (1):

$$\begin{aligned} 2K_0 = {} & [(Ap^2 + Aq^2 + 2z_0\gamma_3) + \mu(A\gamma_1^2 + A\gamma_2^2 + C\gamma_3^2)] - 2\lambda(Ap\gamma_1 + Aq\gamma_2 \\ & + Cr_0\gamma_3) + A(\lambda^2 - \mu)({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2). \end{aligned} \tag{8}$$

The invariant manifolds of the system (7)

$$p - \lambda\gamma_1 = 0, \quad q - \lambda\gamma_2 = 0, \quad z_0 - \lambda C r_0 + [\mu(C - A) + \lambda^2 A]\gamma_3 = 0 \tag{9}$$

give the stationary value to the elements of the above family K_0.

The latter directly follows from the equations

$$\begin{aligned} &\frac{\partial K_0}{\partial p} = A(p - \lambda\gamma_1) = 0, \quad \frac{\partial K_0}{\partial q} = A(q - \lambda\gamma_2) = 0, \\ &\frac{\partial K_0}{\partial \gamma_1} = -\lambda A(p - \lambda\gamma_1) = 0, \quad \frac{\partial K_0}{\partial \gamma_2} = -\lambda A(q - \lambda\gamma_2) = 0, \\ &\frac{\partial K_0}{\partial \gamma_3} = z_0 - \lambda C r_0 + [\mu(C - A) + \lambda^2 A]\gamma_3 = 0. \end{aligned} \tag{10}$$

By equating the derivative of K_0 with respect to λ to zero, we obtain the equation

$$\frac{\partial K_0}{\partial \lambda} = -(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3) + A\lambda({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2) = 0,$$

whence we find

$$\lambda = \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)}.$$

When eliminating λ from equation (8) we obtain the envelope for the family of first integrals (8) of the form

$$\begin{aligned} 2\Lambda_0 = &[(Ap^2 + Aq^2 + 2z_0\gamma_3) + \mu(A\gamma_1^2 + A\gamma_2^2 + C\gamma_3^2)] \\ &- \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)^2}{A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)} - \mu A(\gamma_1^2 + \gamma_2^2 + \gamma_3^2). \end{aligned} \tag{11}$$

Let us separate the invariant manifolds which give a stationary value to the first integral Λ_0. The stationary conditions in this case differ from the stationary conditions of the first integral Λ only by the last equation. For Λ_0 it writes:

$$\begin{aligned} \frac{\partial \Lambda_0}{\partial \gamma_3} = z_0 - &\frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2)} Cr_0 + [\mu(C - A) \\ &+ \frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)^2}{A^2({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2)^2} A]\gamma_3 = 0. \end{aligned} \tag{12}$$

Similarly to the case of equations (5) for Λ, the stationary conditions for Λ_0 coincide with the stationary conditions (10) of the corresponding Λ_0 of the family K_0 of first integrals if the following substitution is performed in these stationary conditions

$$\frac{(Ap\gamma_1 + Aq\gamma_2 + Cr_0\gamma_3)}{A({\gamma_1}^2 + {\gamma_2}^2 + {\gamma_3}^2)} = \lambda.$$

Furthermore, similarly to the case of the Lagrange top in the constant force field, in the case of the enveloping integral there appears a relationship $A\lambda = m$ between the parameter λ and the constant of the integral of squares.

If equations (10) are used to remove $p, \ q, \ \gamma_3$ from the integral of squares, then we obtain the 5th degree equation with respect to λ:

$$\begin{aligned} &A^3\lambda^5 - A^2 m\lambda^4 - 2\mu(C - A)A^2\lambda^3 + A[Cr_0 z_0 - 2\mu(C - A)m]\lambda^2 \\ &+\{A[\mu^2(C - A)^2 - z_0^2] + C^2 r_0^2\mu(C - A)\}\lambda - \mu(C - A)[Cr_0 z_0 \\ &+\mu(C - A)m] = 0. \end{aligned} \tag{13}$$

So, there may be up to five regular precessions (9) on the hypersurface of the integral of squares under a fixed value of m in this integral. If one recalls the relationship $A\lambda = m$ between m and λ, which is imposed on these parameters in the process of constructing the envelope integral, then after writing down the equation (13) of the form

$$\begin{aligned} &A^2\lambda^4(A\lambda - m) - 2\mu(C - A)A^2\lambda^3 + A[Cr_0 z_0 - 2\mu(C - A)m]\lambda^2 + \{A[\mu^2 \\ &\times(C - A)^2 - z_0^2] + C^2 r_0^2\mu(C - A)\}\lambda - \mu(C - A)[Cr_0 z_0 + \mu(C - A)m] = 0 \end{aligned}$$

and taking account of the above relationship, it is possible to obtain

$$ACr_0z_0\lambda^2 + [C^2r_0^2\mu(C-A) - Az_0^2]\mu(C-A)Cr_0z_0 = 0.$$

The latter will determine two values of λ corresponding to precessions, on which the enveloping integral reaches its stationary value. The first of the values of $\lambda_1 = -\mu(C-A)Cr_0(Az_0)^{-1}$ (corresponding to $m_1 = \mu(C-A)Cr_0(z_0)^{-1}$) defines the following value for the directional cosine γ_3:

$$\gamma_3 = \frac{\lambda_1 Cr_0 - z_0}{\mu(C-A) + \lambda_1^2 A} = -\frac{z_0[\mu(C-A)C^2r_0^2 + Az_0^2]}{\mu(C-A)[\mu(C-A)C^2r_0^2 + Az_0^2]}.$$

For $[\mu(C-A)C^2r_0^2 + Az_0^2] \neq 0$ one obtains

$$\gamma_3 = -\frac{z_0}{\mu(C-A)}.$$

This kind of regular precessions is distinguished in the family (9) by the property that there is the axis of permanent rotation on the cone which describes the axis of body symmetry for the given precession. This can readily be seen from the expression for

$$\gamma_3 = \frac{\lambda_1 Cr_0 - z_0}{\mu(C-A) + \lambda_1^2 A} \quad \text{when } \lambda_1 \to 0.$$

In the case when the expression in square brackets is zero we have

$$C^2 r_0^{*2} = -\frac{Az_0^2}{\mu(C-A)} \quad \text{and } \lambda_1^{*2} = -\frac{\mu(C-A)}{A}. \tag{14}$$

The angle γ_3 remains arbitrary in this case. Therefore, for $r_0^* and \lambda_1^*$ defined by formulas (14), the precession may be realized for any constant γ_3.

In case of $\lambda_2 = z_0(Cr_0)^{-1}$ the equation for γ_3 assumes the form:

$$[\mu(C-A) + A\lambda_2^2]\gamma_3 = \lambda_2 Cr_0 - z_0 \quad \text{or} \quad \gamma_3[\mu(C-A) + \frac{Az_0^2}{C^2r_0^2}] = 0.$$

As a result, we have either precessions with $\gamma_3 = 0$ or for $C^2r_0^2 = -Az_0^2(\mu(C-A))^{-1}$, $\lambda_2 = -\mu(C-A)(A)^{-1}$ again, as in the previous case, the precession may be realized for any constant value of γ_3.

Here, similarly to the previous case, the enveloping first integral Λ_0 has given the possibility to separate a subset of special precessions in the family of regular precessions giving the stationary value to K_0.

In conclusion consider the construction of an envelope integral for the family of non-quadratic integrals.

The case of the S.V.Kovalevskaya top In the case of the S.V.Kovalevskaya top on the Delaunay manifold

$$p^2 - q^2 - x_0\gamma_1 = 0, \quad 2pq - x_0\gamma_2 = 0,$$

which is the stationary set for the S.V.Kovalevskaya integral [3]

$$V = (p^2 - q^2 - x_0\gamma_1)^2 + (2pq - x_0\gamma_2) = k^2,$$

the differential equations

$$2\dot{p} = qr, \quad 2\dot{q} = -rp + x_0\gamma_3, \quad \dot{r} = -2pq, \quad \dot{\gamma}_3 = -\frac{q(p^2+q^2)}{x_0}$$

have a one-parameter family of first integrals

$$2L = 4p^2 + r^2 - 2\nu(r\gamma_3 + 2\frac{p(p^2+q^2)}{x_0}) + \nu^2(\gamma_3^2 + \frac{(p^2+q^2)^2}{x_0^2}). \tag{15}$$

Stationary conditions for the function L are

$$\begin{aligned} \frac{\partial L}{\partial p} &= 2(1 - \frac{\nu}{x_0}p)(2p - \frac{\nu}{x_0}(p^2+q^2)) = 0, \\ \frac{\partial L}{\partial q} &= -2\frac{\nu q}{x_0}(2p - \frac{\nu}{x_0}(p^2+q^2)) = 0, \\ \frac{\partial L}{\partial r} &= r - \nu\gamma_3 = 0, \quad \frac{\partial L}{\partial \gamma_3} = -\nu(r - \nu\gamma_3) = 0. \end{aligned} \tag{16}$$

As it is obvious from the structure of the left-hand sides of the latter equations, their solutions are represented by invariant manifolds, which form two one-parameter families:

$$1)\ 2p - \frac{\nu}{x_0}(p^2+q^2)) = 0, \quad r - \nu\gamma_3 = 0,$$
$$2)\ q = 0, \quad p = \frac{\nu}{x_0}, \quad r - \nu\gamma_3 = 0.$$

If the variables $p,\ q,\ \gamma_3$ are eliminated from the integral of squares, narrowed onto the Delaunay invariant manifold,

$$r\gamma_3 + 2\frac{p(p^2+q^2)}{x_0} = m, \tag{17}$$

with the aid equations, which define the 2nd type of invariant manifolds, then we obtain the following relation between m and ν:

$$m = \nu + \frac{x_0^2}{\nu^3}. \tag{18}$$

The latter expression implies that there may be up to four 2nd type invariant manifolds for a fixed m on the surface of the integral level (17). It can be easily verified that for $m^2 < 16x_0/3\sqrt{3}$ all the roots of equation (18) are complex. Consequently, for the corresponding values of constant m, the invariant manifolds which are of interest for us do not exist. In the case of the 1st type of invariant manifolds, the relation between m and ν is simpler: $m = \nu$.

Let us construct the enveloping first integral for the family (15). Having obtained ν from the expression of the derivative of L with respect to ν:

$$\frac{\partial L}{\partial \nu} = r\gamma_3 + 2\frac{p(p^2+q^2)}{x_0} - \nu[\gamma_3^2 + \frac{(p^2+q^2)^2}{x_0^2}] = 0,$$

we have

$$\nu = \frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}.$$

Having substituted this value of ν into expression (15), we obtain the desired enveloping integral for the family (15) of the form

$$2\Lambda_0 = (4p^2 + r^2) - [r\gamma_3 + 2\frac{p(p^2+q^2)}{x_0}]^2[\gamma_3^2 + \frac{(p^2+q^2)^2}{x_0^2}]^{-1}.$$

Let us write down the equations defining the stationary set for the latter integral:

$$\frac{\partial \Lambda_0}{\partial p} = 2[1 - \frac{r\gamma_3 + 2p(p^2+q^2)x_0{}^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}\frac{p}{x_0}][2p - \frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}$$
$$\times \frac{(p^2+q^2)}{x_0}] = 0,$$
$$\frac{\partial \Lambda_0}{\partial q} = -\frac{2q[r\gamma_3 + 2p(p^2+q^2)x_0^{-1}]}{x_0[\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}]}[2p - \frac{(r\gamma_3 + 2p(p^2+q^2)x_0^{-1})}{(\gamma_3^2 + (p^2+q^2)^2 x_0^{-2})}$$
$$\times \frac{(p^2+q^2)}{x_0}] = 0,$$
$$\frac{\partial \Lambda_0}{\partial r} = r - \frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}\gamma_3 = 0,$$
$$\frac{\partial \Lambda_0}{\partial \gamma_3} = -\frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}[r - \frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}}\gamma_3] = 0.$$

Formally, the equations obtained again coincide with equations (16) when the following substitution is performed in them:

$$\frac{r\gamma_3 + 2p(p^2+q^2)x_0^{-1}}{\gamma_3^2 + (p^2+q^2)^2 x_0^{-2}} = \nu.$$

The latter relation and the values of the first integral constants, which occur in this formula, imply the dependence between m and ν:

$$\nu = m.$$

This dependence is incompatible with the conditions of existence of the 2nd type invariant manifolds (18). Therefore, the enveloping first integral has only the 1st type invariant manifolds, whose co-dimension is smaller than that of the 2nd type invariant manifolds, in the capacity of its stationary set.

In conclusion it is expedient to note that for the quite integrable systems, as well as for those close to them, rather typical is the situation, which is considered above on the examples from rigid body dynamics, when a family of first integrals is put in correspondence to a family of IMSMs. In this case, constructing the enveloping first integral not only leads to finding out the peculiarities in the co-dimension of one but also allows one to obtain some additional information on the structure of the respective family of IMSMs.

When the aids of computer algebra are available, the technique proposed allows one to investigate systems of rather high dimensions. For example, the above approach has been employed for finding out peculiar IMSMs of a system of rigid bodies with the carrier [4], whose lagrangian writes:

$$2L = \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} J_{\alpha\beta}(q)w_\alpha w_\beta + 2\sum_{k=1}^{n}\sum_{\alpha=1}^{3} e_{k\alpha}(q)w_\alpha \dot{q}_k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{q}_i\dot{q}_j + 2U(q), \tag{19}$$

where w_α $(\alpha = 1,2,3)$are the projections of the carrier body's angular rate onto the axes bound up with this body; $q_i, \dot{q}_i$ $(i = 1,\dots,n)$ are the generalized coordinates and velocities which determine the orientation of the carried bodies with respect to one another and to the carrier's body; $U(q_1,\dots,q_n)$ is the force function.

According to [4], the system of differential equations, which is defined by the lagrangian (19) and complemented with Poisson's equations for the directional cosines, assumes the one-parameter family of first integrals:

$$\begin{aligned}
2K_2 = {} & \left[\sum_{\alpha=1}^{3} J_{1\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k1}\dot{q}_k\right]^2 + \left[\sum_{\alpha=1}^{3} J_{2\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k2}\dot{q}_k\right]^2 \\
& + \left[\sum_{\alpha=1}^{3} J_{3\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k3}\dot{q}_k\right]^2 - 2\lambda\left\{\left[\sum_{\alpha=1}^{3} J_{1\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k1}\dot{q}_k\right]\gamma_1\right. \\
& \left. + \left[\sum_{\alpha=1}^{3} J_{2\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k2}\dot{q}_k\right]\gamma_2 + \left[\sum_{\alpha=1}^{3} J_{3\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k3}\dot{q}_k\right]\gamma_3\right\} \\
& + \lambda^2[\gamma_1^2 + \gamma_2^2 + \gamma_3^2].
\end{aligned} \tag{20}$$

Here γ_i $(i = 1,2,3)$ are the directional cosines for the "vertical" in the axes bound up with the carrier's body.

It can readily be verified – by writing out the necessary conditions of extremum for K_2 – that the stationary value for the elements of the family of first integrals K_2 may be provided by the elements of the family of IMSMs defined by the following equations:

$$\begin{aligned}
&\sum_{\alpha=1}^{3} J_{1\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k1}\dot{q}_k - \lambda\gamma_1 = 0, \\
&\sum_{\alpha=1}^{3} J_{2\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k2}\dot{q}_k - \lambda\gamma_2 = 0, \\
&\sum_{\alpha=1}^{3} J_{3\alpha}(q)w_\alpha + \sum_{k=1}^{n} e_{k3}\dot{q}_k - \lambda\gamma_3 = 0.
\end{aligned} \tag{21}$$

Here, similarly to the above examples for the Lagrange and Kovalevskaya tops, an envelope for the family of first integrals (20) has been constructed, and with the aid of the first integral thus obtained the peculiar IMSMs from the family (21) have been found. The computations have been conducted for a fixed n. The results of the investigation will not be given since they are rather bulky.

References

1. Borisov, A.V., Matveyev, I.S.: Poisson Structures and Lie Algebras in Hamiltonean Mechanics. "Udmurtiya University" Publ., 1999 (in Russian)

2. Beletsky, V.V.: Motion of an Artificial Satellite with Respect to the Mass Center. Moscow, Nauka Publ., 1965 (in Russian)
3. Irtegov, V.D.: Invariant Manifolds of Steady Motions and Their Stability. Novosibirsk, Nauka Publ., 1985 (in Russian)
4. Litvyn-Sedoy, M.Z.: Mechanics of Systems of Interconnected Rigid Bodies. Moscow: VINITI. Itogy Nauky y Techniky. General Mechanics. Vol. 5, 1982, 3–61 (in Russian)