On Asymptotic Solutions of Higher-Order Boundary Value Problems

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Abstract. This paper presents an algorithm for finding asymptotic solutions to singularly perturbed higher-order boundary value problems. The main idea is to regard an n-th order problem as the combination of an initial value problem and a singularly perturbed second order boundary value problem for which an extensive theory exists. An algorithm for finding asymptotic solutions is implemented in *Mathematica*. It is illustrated by two worked examples.

1 Introduction

Higher order boundary value problems (of order greater than two) occur in different applications. For example, the motion of thin liquid films subject to viscous, capillary and gravitational forces is governed by the third-order boundary value problem. Other viscocapillary flows such as horizontal coating flow, draining down a dry vertical wall, or liquid film falling on a vertical wall are just a few other examples ([1], [2], [8]). Many of such models are singular perturbation problems: the highest derivative is multiplied by a small parameter, ϵ.

Numerical solutions to such singularly perturbed higher-order problems are not easy to find. Existing software packages for solving boundary value problems fail for problems with sharp boundaries or thin internal layers unless the mesh is accurately initialised. An effective MATLAB solver, *bvp4c* ([11]), cannot find a solution to a simple third-order boundary value problem

$$\begin{array}{l} \epsilon y''' = {y''}^2 - 1,\ 0 < t < 1 \\ y(0) = A_0, y'(0) = A_1, y'(1) = B_1 \end{array}$$

for $\epsilon \leq 0.01$ unless a good starting guess describing qualitative behaviour of the system is provided either by the user or by the results of running *bvp4c* with higher values of ϵ. At best, numerical solver will yield a single solution even for a multi-valued problem.

To determine whether a boundary value problem has a single solution, multiple or infinitely many solutions, and to construct these solution(s) an analytical study of the system is required. This is the case for the second order problems as well as for the higher-order ones. To author's knowledge, there have been few attempts so far to automate in Computer Algebra existing theoretical results regarding finding asymptotics to singularly perturbed problems. One of examples, is a symbolic-numerical approach to solving semilinear second-order systems ([10]) based on the theory from ([9]).

This paper considers a simple and direct approach for finding asymptotic solutions to higher order boundary value problems based on the classical theory for second-order problems and differential inequalities of higher order ([4]). An algorithm for tackling higher-order problems is considered and implemented in *Mathematica* by the author. Algorithm provides conditions for the existence of solutions, and constructs multiple piece-wise solutions for higher order boundary value problems. Such asymptotic solutions describe the system behaviour reasonably well. In the case when more accuracy is needed, asymptotic solutions obtained from the algorithm can be used as a starting guess for the numerical solver.

2 Singular Perturbation Problems of Higher Order

Consider an n-th order ($n \geq 3$) scalar singularly perturbed boundary value problem $0 < \epsilon \ll 1$):

$$\begin{array}{l} \epsilon y^{(n)}(t) = f(t, y, y', y'', \ldots, y^{(n-1)}),\ a < t < b \\ y^{(j)}(a) = A_j,\ 0 \leq j \leq (n-2) \end{array}$$

$$y^{(n-2)}(b) = B_{n-2}. \tag{1}$$

Under certain conditions outlined below, the asymptotic solution to such boundary value problem exists and can be constructed ([4]). The main idea is to regard the given problem as the combination of a singularly perturbed second-order problem and an unperturbed initial value problem. Let us rewrite n-th order equation (1) as the following system:

$$\begin{aligned} y_i' &= y_{i+1}, \quad y_i(a;\epsilon) = A_{i-1}, \; i = 1 \dots n-3 \\ y_{n-2}' &= z, \quad y_{n-2}(a;\epsilon) = A_{n-3} \\ \epsilon z'' &= f(t, y_1, \dots, y_{n-2}, z, z'), \\ z(a;\epsilon) &= A_{n-2}, \; z(b;\epsilon) = B_{n-2}. \end{aligned} \tag{2}$$

Asymptotic behaviour of scalar n-th order equation (1) is determined by the n-th order system (2), or more precisely by its reduced system

$$\begin{aligned} Y_i' &= Y_{i+1}, \quad Y_i(a;\epsilon) = A_{i-1}, \; i = 1 \dots n-3 \\ Y_{n-2}' &= z, \quad Y_{n-2}(a;\epsilon) = A_{n-3} \\ 0 &= f(t, Y_1, \dots, Y_{n-2}, Z, Z'), \\ Z(a;\epsilon) &= A_{n-2}, \; Z(b;\epsilon) = B_{n-2}. \end{aligned} \tag{3}$$

When viewed in this light, the behaviour of the n-th order boundary value problem (1) is determined by the behaviour of the second-order problem:

$$\begin{aligned} \epsilon z''(t) &= \mathbf{F}(t, z, z'), \; a < t < b \\ z(a) &= A_{n-2}, \; z(b) = B_{n-2}, \\ \mathbf{F}(t, z, z') &= f(t, Y_1, Y_2, \dots, Y_{(n-2)}, z, z'). \end{aligned} \tag{4}$$

For problems of this type there are extensive theoretical results regarding existence of asymptotic solutions and their construction (e.g. [3], [4], [5]).

If the right-hand side of original equation (1) depends on $y^{(n-2)}$ but does not depend on $y^{(n-1)}$, the corresponding system (4) is *semilinear*

$$\begin{aligned} \epsilon z''(t) &= \mathbf{F}(t, z), \; a < t < b \\ z(a) &= A_{n-2}, \; z(b) = B_{n-2}, \\ \mathbf{F}(t, z) &= f(t, Y_1, Y_2, \dots, Y_{(n-2)}, z). \end{aligned} \tag{5}$$

To solve such problem a theory of semilinear problems ([9], [10]) can be employed. These results are not considered in detail in this paper. If the right-hand side of the original problem (1) does not depend on either $y^{(n-1)}$ and $y^{(n-2)}$, the corresponding system (4) is linear

$$\begin{aligned} \epsilon z''(t) &= \mathbf{F}(t), \; a < t < b \\ z(a) &= A_{n-2}, \; z(b) = B_{n-2}, \\ \mathbf{F}(t) &= f(t, Y_1, Y_2, \dots, Y_{(n-2)}). \end{aligned} \tag{6}$$

Special cases of *quasilinear*

$$f = y^{(n-1)} + g(t, y, y', \dots y^{(n-2)})$$

and *quadratic*

$$f = (y^{(n-1)})^2 + g(t, y, y', \dots y^{(n-2)})$$

dependencies can also be considered ([3]).

This paper discusses an algorithm for a general class of nonlinear boundary value problems. This is the case when the right-hand side in the original problem (1) depends on $y^{(n-1)}$, and the corresponding second-order system (4) is nonlinear. To illustrate the approach outlined above, consider the case when the second order boundary value problem has *angular* solutions.

3 Theory for 2nd-Order Boundary Value Problems

Consider the second order boundary value problem:

$$\begin{aligned} \epsilon y'' &= F(t, y, y'), \; a < t < b \\ y(a) &= A, \quad y(b) = B. \end{aligned} \tag{7}$$

3.1 Angular Solutions

The classical Haber-Levinson theory states that if the assumptions below are satisfied, the system (7) has a locally unique solution. Suppose that the corresponding reduced equation

$$\begin{aligned} 0 &= F(t, y, y'), \;\; a < t < b \\ y(a) &= A, \;\; y(b) = B \end{aligned} \tag{8}$$

has a left solution u_L valid on $[a, t_L]$ and a right solution u_R valid on $[t_R, b]$, such that

$$u_L(a) = A, \;\; u_R(b) = B. \tag{9}$$

Intersection condition. The solutions u_L and u_R are called *angular* if they intersect at a point $t_0 \in [a, b]$ with unequal slopes

$$\begin{aligned} u_L(t_0) &= u_R(t_0) = \sigma_0 \\ \mu_L = u'_L(t_0) &\neq u'_R(t_0) = \mu_R. \end{aligned} \tag{10}$$

Stability condition. The solutions u_L and u_R are stable in the sense that

$$\begin{aligned} F_{y'}(t, u_L(t), u'_L(t)) &\geq k > 0, \;\; a < t < t_0 \\ F_{y'}(t, u_R(t), u'_R(t)) &\leq -k < 0, \;\; t_0 < t < b. \end{aligned} \tag{11}$$

Crossing condition holds for all λ between μ_L and μ_R

$$(\mu_R - \mu_L) F(t_0, u_L(t_0), \lambda) > 0. \tag{12}$$

Then, the original problem has a solution

$$y(t) = \begin{cases} u_L(t) + \xi_L, & a \leq t < t_0, \\ u_R(t) + \xi_R, & t_0 < t \leq b. \end{cases} \tag{13}$$

where ξ_L and ξ_R are asymptotic angular corrections to the left and to the right of the turning point t_0

$$\begin{aligned} \xi_L &= \frac{\epsilon}{k}(\mu_L - \mu_R)\exp\{-\frac{k}{\epsilon}(t_0 - t)\}, \;\; a \leq t < t_0 \\ \xi_R &= \frac{\epsilon}{k}(\mu_R - \mu_L)\exp\{-\frac{k}{\epsilon}(t - t_0)\}, \;\; t_0 < t \leq b. \end{aligned} \tag{14}$$

Remarks: Needless to say that solving reduced equation (8) analytically is not always possible. However, solving a reduced equation numerically is much easier that solving the original problem. The reason being that the problems (8) for u_L and u_R with the respective condition (9) are initial value problems of order one while the original problem (7) is a singularly perturbed boundary value problem of order two.

3.2 Double Crossings.

Results similar to the above hold if reduced equation has more than two solutions on $[a, b]$. Consider the case of three solutions. u_L, u_M and u_R defined on $[a, t_1]$, $[t_1, t_2]$ and $[t_2, b]$ respectively such that

$$\begin{aligned} u_L(a) = A, \;\; u_L(t_1) &= u_M(t_1), \\ u_M(t_2) = u_R(t_2), \;\; u_R(b) &= B. \\ \mu_L = u'_L(t_1) \neq u'_M(t_1) &= \mu_{M1}, \\ \mu_{M2} = u'_M(t_2) \neq u'_R(t_2) &= \mu_R. \end{aligned} \tag{15}$$

Stability conditions. Sign restrictions for left and right solutions must hold on the whole intervals, while sign restrictions for the middle solution must only hold in a small neighbourhood of a turning point:

$$\begin{aligned} F_{y'}(t, u_L(t), u'_L(t)) &\geq k_1 > 0, \;\;\; a \leq t < t_1 \\ F_{y'}(t, u_M(t), u'_M(t)) &\leq -k_1 < 0, \;\;\; t_1 < t < t_1 + \delta \\ F_{y'}(t, u_M(t), u'_M(t)) &\geq k_2 > 0, \;\;\; t_2 - \delta \leq t < t_2 \\ F_{y'}(t, u_R(t), u'_R(t)) &\leq -k_2 < 0, \;\;\; t_2 < t \leq b. \end{aligned} \tag{16}$$

Crossing conditions hold strictly for λ between μ_L and μ_{M1}

$$(\mu_{M1} - \mu_L)F(t_1, u_L(t_1), \lambda) > 0 \tag{17}$$

and for λ between μ_{M2} and μ_R

$$(\mu_{M2} - \mu_R)F(t_2, u_R(t_2), \lambda) > 0, \tag{18}$$

Stability condition for middle solution. The middle solution is stable in the sense that there exists a positive constant l such that

$$F_y(t, u_M(t), u'_M(t)) \geq l > 0, \ t_1 + \delta \leq t \leq t_2 - \delta \tag{19}$$

Then, the original problem has a solution $y(t)$

$$y(t) = \begin{cases} u_L(t) + \xi_L, & a \leq t < t_1, \\ u_M(t) + \xi_{M1} + \xi_{M2}, & t_1 < t \leq t_2 \\ u_R(t) + \xi_R, & t_2 < t \leq b. \end{cases} \tag{20}$$

where ξ_L, ξ_M, and ξ_R are asymptotic angular corrections defined as in (14). This result can be generalised for any finite number of crossings.

3.3 Modified Stability

Some modifications of the conditions under which the system (7) has angular solutions have been considered in ([5]). For instance, a boundary value problem (7) will still have an angular asymptotic solution if the requirement of $F_{y'}$ stability (11) is substituted by the requirement of F_y stability (22) provided $F_{y'}$ stability in a small neighbourhood of a turning point t_0 holds:

$$\begin{aligned} F_{y'}(t, u_L(t), u'_L(t)) \geq k > 0, \ t_0 - \delta < t < t_0 \\ F_{y'}(t, u_R(t), u'_R(t)) \leq -k < 0, \ t_0 < t < t_0 + \delta \end{aligned} \tag{21}$$

F_y stability condition is given by

$$\begin{aligned} F_y(t, u_L(t), u'_L(t)) \geq l > 0 \quad a < t < t_0 - \delta \\ F_y(t, u_R(t), u'_R(t)) \geq l > 0 \quad t_0 + \delta < t < b. \end{aligned} \tag{22}$$

This result holds for function f which satisfies

$$|F(t, y, y')| \to \mathrm{O}(|y'|) \text{ as } |y'| \to \infty. \tag{23}$$

4 Theory for Higher Value Boundary Value Problems

When the n-th order boundary value problem

$$\begin{aligned} \epsilon y^{(n)}(t) &= f(t, y, y', y'', \ldots, y^{(n-1)}), \ a < t < b \\ y^{(j)}(a) &= A_j, \ 0 \leq j \leq (n-2) \\ y^{(n-2)}(b) &= B_{n-2} \end{aligned}$$

is viewed as a system of initial value problem and a second order boundary value problem (2), the main results concerning asymptotic solutions can be extended to higher order case. Let us now formulate conditions under which the n-th order problem has an asymptotic angular solution.

4.1 Interior Layer Behaviour

Let the corresponding reduced equation have a left solution u_L valid on $[a, t_L]$ and a right solution u_R valid on $[t_R, b]$ such that

$$\begin{aligned} u_L^{(j)}(a) &= A_j, \ j = 0 \ldots n-2 \\ u_R^{(n-2)}(b) &= B_{n-2}. \end{aligned} \tag{24}$$

As is the case for the second-order problems, solving the reduced equation (3) of order $n-1$ analytically is not always possible. However, obtaining numerical solution to an initial value problem of order $n-1$ is easier than solving the original n-order boundary value problem.

Intersection condition. Two solutions of higher order value problem are called angular if they intersect at a point $t_0 \in [a, b]$ so that

$$\begin{aligned} u_L^{(j)}(t_0) = u_R^{(j)}(t_0) = \sigma_j,\ j = 0 \dots n-2 \\ \mu_L = u_L^{(n-1)}(t_0) \neq u_R^{(n-1)}(t_0) = \mu_R. \end{aligned} \tag{25}$$

Stability condition. For n-order problems, the stability is determined with respect to the highest derivative on the right-hand side. The solutions u_L and u_R are said to be stable if

$$\begin{aligned} f_{y^{(n-1)}}(u_L, u_L', \dots, u_L^{(n-1)}) \geq k > 0,\ a \leq t \leq t_0 \\ f_{y^{(n-1)}}(u_R, u_R', \dots, u_R^{(n-1)}) \leq -k > 0,\ t_0 \leq t \leq b. \end{aligned} \tag{26}$$

Crossing condition. The crossing condition holds for λ between μ_L and μ_R

$$(\mu_R - \mu_L) f(t_0, \sigma_0, \sigma_1, \dots \sigma_{n-2}, \lambda) > 0 \tag{27}$$

Then, the original problem (1) has a solution $y(t;\epsilon)$ such that

$$\begin{aligned} y^{(j)} &= Y^{(j)} + O(\epsilon),\ j = 0 \dots n-3 \\ y^{(n-2)} &= \begin{cases} u_L^{(n-2)}(t) + \xi_L, & a \leq t < t_0, \\ u_R^{(n-2)}(t) + \xi_R, & t_0 < t \leq b. \end{cases} \end{aligned} \tag{28}$$

where ξ_L and ξ_R are asymptotic angular corrections defined by equation (14).

4.2 Shock Layer Behaviour

Solutions of (1) may also exhibit shock layer behaviour when the left and right solutions of the reduced equation exist but do not satisfy intersection conditions (25). Namely,

$$u_L^{(n-2)}(t_0) \neq u_R^{(n-2)}(t_0). \tag{29}$$

Theory for shock solutions to nonlinear problems is restricted to quasilinear and semilinear problems. Consider a quasilinear problem wherein the right-hand side of equation (1) is given by

$$f = h(t, y, y' \dots y^{n-2}) y^{(n-1)} + g(t, y, y' \dots, y^{(n-2)})$$

The point of shock, t_0, is determined from

$$J(t) = \int_{u_R}^{u_L} h(s, u, u', \dots u^{(n-1)}) ds,\ J(t_0) = 0. \tag{30}$$

Free constants in the right solutions are determined from intersection conditions

$$u_L^{(j)}(t_0) = u_R^{(j)}(t_0) = \sigma_j,\ j = 0 \dots n-3. \tag{31}$$

In this case, $(n-2)$ derivatives transfer from one reduced solution to another discontinuously at a point $t = t_0$ with the asymptotic corrections are given by

$$\begin{aligned} \xi_{sL} = \frac{1}{2}(u_R^{(n-2)}(t_0) - u_L^{(n-2)}(t_0)) \exp\{\frac{k}{\epsilon}(t - t_0)\} \\ \xi_{sR} = \frac{1}{2}(u_L^{(n-2)}(t_0) - u_R^{(n-2)}(t_0)) \exp\{\frac{k}{\epsilon}(t_0 - t)\}. \end{aligned} \tag{32}$$

Here k is determined from the stability conditions:

$$\begin{aligned} h(t, u, u', \dots, u^{(n-2)}) \geq k > 0, a \leq t < t_0 \\ h(t, u, u', \dots, u^{(n-2)}) \leq -k < 0, t_0 < t \leq b. \end{aligned} \tag{33}$$

System representation (2) can also be employed for the cases when reduced equation (3) does not have closed-form solutions.

4.3 Semilinear Problems

Consider briefly the semilinear case when function f in the right-hand side of (1) is independent of $y^{(n-1)}$. The solution to this problem exists if the reduced problem has two angular solutions $u_L(t)$ and $u_R(t)$ which satisfy (25) and crossing condition (27). In addition, f is stable in the sense that

$$f_{y^{(n-2)}}(u, u', \dots, u^{(n-2)}) \geq k > 0 \quad [a, b] \tag{34}$$

Then, the asymptotic angular correction to the $y^{(n-2)}$ are given by

$$\begin{aligned} \xi_L(t;\epsilon) &= \sqrt{\frac{\epsilon}{k}}(\mu_R - \mu_L)\exp\{-\sqrt{\frac{\epsilon}{k}}(t_0 - t)\} \\ \xi_R(t;\epsilon) &= \sqrt{\frac{\epsilon}{k}}(\mu_L - \mu_R)\exp\{-\sqrt{\frac{\epsilon}{k}}(t - t_0)\}. \end{aligned} \tag{35}$$

In the semilinear case, solution can exhibit shock layer behaviour where the point of shock is determined from (30) by changing h to f. Shock layer corrections are given by equations (32). The case of semilinear systems have been considered in detail in ([9], [10]).

4.4 Boundary Layer Behaviour

Consider now boundary value behaviour. Note, that for functions f of the type $|f(t, y, y')| = \mathrm{O}(|y'|^n)$, $n > 2$, $y' \to \infty$ the boundary layer behaviour is not possible ([7]). Suppose $u_L(t)$ is the solution to the reduced problem (3) which satisfies initial conditions

$$y^{(j)}(a) = A_j, \ j = 0 \dots n-2. \tag{36}$$

Function f is continuous with respect to $(t, y, y' \dots, y^{(n-1)})$ and grows at most linearly as a function of $y^{(n-1)}$:

$$|f(t, y, y' \dots y^{n-2})| = \mathrm{O}(|y^{(n-1)}|) \text{ as } y^{(n-1)} \to \infty. \tag{37}$$

In addition, the derivative is required to be positive along the solution of the reduced problem:

$$f_{y^{(n-1)}}(t, u_L, u_L', \dots, u_L^{(n-1)}) \geq k > 0, \quad a \leq t \leq b. \tag{38}$$

Then, the original problem (1) has a solution $y(t;\epsilon)$ which exhibits boundary layer behaviour. Asymptotic correction to $y^{(n-2)}$ is given by

$$\xi_{BR}(t;\epsilon) = (B_{n-2} - u_L^{(n-2)}(b))\exp\{-\frac{k}{\epsilon}(b - t)\}. \tag{39}$$

In the case of quadratic dependence of function f on $y^{(n-1)}$

$$|f(t, y, y', \dots, y^{(n-1)})| = O(|y^{(n-1)}|^2) \text{ as } y^{(n-1)} \to \infty \tag{40}$$

solution exhibits boundary layer behaviour if there exists a positive constant $k > 0$ such that

$$\frac{\partial f}{\partial y^{(n-1)}} \geq 0, \ \frac{\partial f^2}{\partial^2 y^{(n-1)}} \geq k > 0. \tag{41}$$

Thus, the asymptotic boundary layer correction is given by

$$\begin{aligned} \xi_{2BR}(t;\epsilon) = k\epsilon \ln[(b-a)^{-1}(b - t + \\ (t-a)\exp\{-(\epsilon/k)|B_{n-2} - u_L^{(n-2)}(b)|\} \end{aligned} \tag{42}$$

Given the above results for interior and boundary layers, let us now formulate an algorithm for finding asymptotic solutions to higher order singularly perturbed boundary value problems.

5 Algorithm

- Find the order of equation, n.
- Determine the type of problem: nonlinear, semilinear, quasilinear, quadratic, linear.
- Solve two initial value problems

$$y^{(k)} = g(t, y, y', \dots, y^{(l)}),\ l \leq k,\ y^{(j)} = A_j \tag{43}$$

  and

$$y^{(k)} = g(t, y, y', \dots, y^{l}),\ l \leq k,\ y^{(j)} = B_j \tag{44}$$

  where $y^{(k)} = g(t, y, y', \dots, y^{l})$ is a solution of an algebraic equation $f(t, y, y\prime, \dots, y^{(k)}) = 0$ with respect to its highest derivative $y^{(k)}$. The solutions to the right problem (44) contain unknown constants to be determined later.
- If a closed-form solution to the initial value problem (43) is not found, an algorithm can be adapted to incorporate numerical part.
- Select stable solutions and find their intervals of stability. Depending on the problem type different types of stability is investigated ($f_{y^{(n-1)}}$ or $f_{y^{(n-2)}}$). In the case of a nonlinear equation, left solution $u_L(t)$ is said to be stable if

$$\frac{\partial f}{\partial y^{(n-1)}}|_{y=u_L(t)} \geq k > 0, u_L(a) = A$$

  and right solution is stable if

$$\frac{\partial h}{\partial y^{(n-1)}}|_{y=u_R(t)} \leq -k < 0, u_R(a) = B.$$

  For the semilinear type of problem, a solution $u(t)$ is said to be stable if

$$\frac{\partial f}{\partial y^{n-2}}|_{y=u(t)} \geq k > 0.$$

- Find minimums of corresponding partial derivatives.
- Find the type of interior crossings: if left and right solutions satisfy condition (14), the crossing is angular. Shock layer behaviour is only possible for quasilinear and semilinear problems. In the latter case, the point of shock is determined from (30).
- Check that crossing conditions (27) are satisfied.
- Construct a piece-wise path from $t = a$ to $t = b$ from stable pieces of solutions using algorithm developed for semilinear systems ([10]).
- Construct asymptotic solutions by adding corresponding asymptotic corrections of the correct type to the path.

6 Examples

An algorithm for finding asymptotic solutions to higher-order boundary value problems has been implemented *Mathematica* by the author. Let us consider two simple examples whose solutions display behaviour outlined above.

7 Examples

7.1 Example 1

Consider a third-order boundary value problem:

$$\begin{aligned} \text{example1} = \epsilon y'''(t) = (y'')^2 - 1 \\ y(0) = 1, y'(0) = 1/2, y'(1) = 1 \end{aligned} \tag{45}$$

HigherOrderEqns[example1, y[t], t, 0, 1]
Third order nonlinear quadratic equation

Solving Reduced Equation...
raw solutions:

$$\{\{y[t]->\frac{t^2}{2}+C[1]+tC[2],y[t]->-\frac{t^2}{2}+C[1]+tC[2]\}\}$$

$$\begin{aligned}&\textit{stable left solutions}:\{\{1+\frac{t}{2}+\frac{t^2}{2}\}\}\text{ on }[0,1]\\&\textit{stable right solutions}:\{\{C[1]+t-\frac{t^2}{2}\}\}\text{ on }[0,1]\end{aligned}$$

angular intersection

$$t=\frac{1}{4},\ C[1]=\frac{15}{16}$$

crossing conditions holds true
asymptotic solutions with boundary layer at $t=1$

$$\begin{aligned}u=&\,1+\frac{t}{2}+\frac{t^2}{2}-2t\epsilon\\&+\frac{2\left(1-t+e^{-\frac{3}{4\,\epsilon}}\,t\right)\,\epsilon\,\log\left[1-t+e^{-\frac{3}{4\,\epsilon}}\,t\right]}{-1+e^{-\frac{3}{4\,\epsilon}}}\end{aligned}$$

angular solution

$$\begin{aligned}uL&=1+\frac{t}{2}+\frac{t^2}{2}-\frac{1}{2}\,e^{-\frac{2t-1}{2\,\epsilon}}\,\epsilon^2,\ 0\le t<\frac{1}{4}\\uR&=\frac{15}{16}+t-\frac{t^2}{2}+\frac{1}{2}\,e^{-\frac{1-2t}{2\,\epsilon}}\,\epsilon^2\frac{1}{4}<t\le 1\end{aligned}$$

Boundary value problem (45) was attempted to be solved numerically by *bvp4c* solver ([11]). For $\epsilon <= 0.01$ the solver fails to converge. The continuation technique when a result from the problem with a higher value of ϵ is used as a starting guess for smaller ϵ yields only one solution rather than two solutions found by an algorithm (see Figure 1).

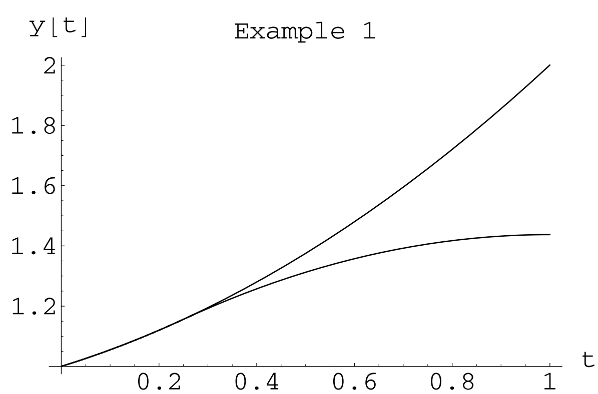

Fig. 1. Two solutions to example1 (45): a solution with a boundary layer at $t=1$ and an angular solution with intersection point $t=1/4$. $\epsilon=0.01$.

7.2 Example 2

Consider now an example with a shock layer behaviour:

$$\begin{aligned} \text{example2} = \epsilon y'''[t] &= y'[t] - y'[t]y''[t] \\ y[0] &= 1, y'[0] = 1/2, y'[1] = 1 \end{aligned} \tag{46}$$

HigherOrderEqns[example2, y[t], t, 0, 1]
Third order quasilinear equation
Solving Reduced Equation
raw solutions:

$$\{\{y[t]->C[1], y[t]->-\frac{t^2}{2}+C[1]+tC[2]\}\}$$

$$\textit{stable left solutions} :\{\{1-\frac{t}{2}+\frac{t^2}{2}\}\} \text{ on } [0,\frac{1}{2}]$$
$$\textit{stable right solutions} :\{\{C[1]+\frac{t^2}{2}\}\} \text{ on } [0,1]$$

shock at

$$t=\frac{1}{4},\ C[1]=\frac{7}{8}$$

asymptotic solutions: shock layer behaviour

$$uL = 1-\frac{t}{2}+\frac{t^2}{2}+\frac{\epsilon}{8}-\frac{1}{8}\,e^{-\frac{1}{4\,\epsilon}}\,\epsilon,\ 0\le t<\frac{1}{4}$$
$$uR = \frac{7}{8}+\frac{t^2}{2}-\frac{\epsilon}{8}+\frac{1}{8}\,e^{-\frac{1}{4\,\epsilon}}\,\epsilon,\ \frac{1}{4}<t\le 1.$$

An asymptotic solution with a shock layer is presented on Figure 2.

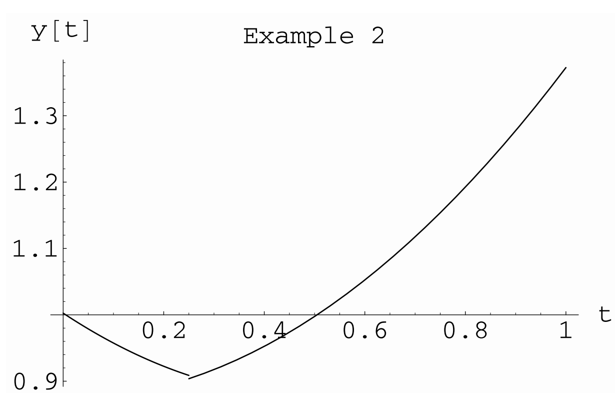

Fig. 2. Asymptotic solution to example2 (46) with a shock layer at $t = 1/4$ is shown. $\epsilon = 0.005$.

8 Summary

This paper discusses finding asymptotic solutions for higher order boundary value problems. The algorithm presented in this paper determines the leading asymptotic solution which can either be found in a fully analytical form or a symbolic-numerical one. In both cases, the task of finding a solution to a higher order boundary value problem is simplified to solving an initial value problem of an order lower than the one for the original problem and the second-order boundary value problem. For many problems, asymptotics is close enough to exact solution. If more accuracy is needed, asymptotics can then be used as an initial guess for the boundary value solver.

The current algorithm can be applied to general nonlinear type of higher-order boundary value problem. The algorithm can easily be extended to systems of higher-order initial boundary value problems. Cases of generalised and weak stability should also be added. In cases, when the closed-form solution of reduced solution cannot be found, a numerical method based on the presented algorithm can be envisaged.

The boundary conditions for the considered problems have been of Dirichlet type. The algorithm can be extended to non-Dirichlet types of separate boundary conditions. Asymptotic solutions to problems with Robin or Neumann type of boundary conditions

$$\begin{aligned} y^{(j)} &= A_i, \ i = 0, \ldots, n-3 \\ p_1 y^{(n-2)}(a) - q_1 y^{(n-1)}(a) &= A_{n-2}, \ p_1 + q_1 > 0 \\ p_2 y^{(n-2)}(b) + q_2 y^{(n-1)}(b) &= B_{n-2}, \ p_2 + q_2 > 0 \end{aligned}$$

can be found in a similar way applying the results for the second order problems from ([6]).

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