Computation of Cohomology of Lie Algebra of Hamiltonian Vector Fields by Splitting Cochain Complex into Minimal Subcomplexes

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Abstract. Computation of homology or cohomology is intrinsically a problem of high combinatorial complexity. Recently we proposed a new efficient algorithm for computing cohomologies of Lie algebras and superalgebras. This algorithm is based on partition of the full cochain complex into minimal subcomplexes. The algorithm was implemented as a C program **LieCohomology**. In this paper we present results of applying the program **LieCohomology** to the algebra of hamiltonian vector fields H(2|0). We demonstrate that the new approach is much more efficient comparing with the straightforward one. In particular, our computation reveals some new cohomological classes for the algebra H(2|0) (and also for the Poisson algebra Po(2|0)).

1 Introduction

Cohomology is defined by *cochain complex*

$$0 \to C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots. \quad (1)$$

Here C^k are linear spaces (more generally, abelian groups), graded by the integer number k, called *dimension* or *degree* (depending on the context). The elements of the spaces C^k are called *cochains.*

The linear mappings d^k are called *differentials* (or *coboundary operators*). The main property of these mappings is "their squares are equal to zero": $d^k \circ d^{k-1} = 0$.

The elements of the space $Z^k = \mathrm{Ker}\ d^k$ are called *cocycles.* The elements of the space $B^k = \mathrm{Im}\ d^{k-1}$ are called *coboundaries.* Note that $B^k \subseteq Z^k$.

The kth *cohomology* is the quotient space

$$H^k = Z^k/B^k \equiv \mathrm{Ker}\ d^k/\mathrm{Im}\ d^{k-1}.$$

There are many cohomological theories designed for investigation of different mathematical structures and the space H^k carries important information about peculiarities in these structures. The only difference between cohomological theories lies in the constructions of the cochain spaces and coboundary operator. These constructions depend on the underlying mathematical structures.

The cohomology of the Lie (super)algebra A in the module X is defined via cochain complex (1) in which (see, e.g., [1]) the cochain spaces $C^k = C^k(A; X)$ consist of super skew-symmetric k-linear mappings $A \times \cdots \times A \to X$, $C^0 = X$ by definition. Super skew-symmetry means symmetry with respect to swapping of two adjacent *odd* cochain arguments and antisymmetry for any other combination of parities for adjacent pair.

The differential d^k takes the form[1]

$$(d^k c)(a_0, \ldots, a_k) = -\sum_{0 \le i < j \le k} (-1)^{s(a_i)+s(a_j)+p(a_i)p(a_j)} c([a_i, a_j], a_o, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_k)$$
$$- \sum_{0 \le i \le k} (-1)^{s(a_i)} a_i c(a_o, \ldots, \widehat{a}_i, \ldots, a_k), \quad (2)$$

where the functions $c(\ldots)$ are elements of cochain spaces; $a_i \in A$; $p(a_i)$ is the parity of a_i; $s(a_i) = i$, if a_i is even element and $s(a_i)$ is equal to the number of even elements in the sequence $a_0, \ldots, a_{i-1}$, if a_i is

[1] This version of formula for differential corresponds to the algorithm used in the program **LieCohomology**.

odd element. In the case of trivial module (i.e., if $ax = 0$ for all $a \in A$ and $x \in X$) one uses as a rule the notation $H^k(A)$.

In papers [2–5] we presented an algorithm for computation of Lie (super)algebra cohomologies. These papers contain also the description of its C implementation and some results obtained with the help of codes designed. This algorithm computes cohomology of Lie (super)algebra A over module X in a straightforward way, i.e., for cochain complex (1) the algorithm constructs the full set of basis super skew-symmetric monomials forming the space C^k, generates subsequently all basis monomials in the space C^{k+1}, computes the differentials corresponding to these monomials to obtain the set of linear equations determining the space of cocycles

$$Z^k = \mathrm{Ker}\ d^k = \{C^k \mid dC^k = 0\}, \tag{3}$$

constructs the space of coboundaries

$$B^k = \mathrm{Im}\ d^{k-1} = \{C^k \mid C^k = dC^{k-1}\}. \tag{4}$$

Finally, the algorithm constructs the basis elements of quotient space

$$H^k(A; X) = Z^k/B^k. \tag{5}$$

This last step is based on the Gauss elimination procedure.

The main difficulty in computing cohomology results from the very high dimensions of the spaces C^k: for n-dimensional ordinary Lie algebra and p-dimensional module

$$\dim C^k = p\binom{n}{k},$$

and for $(n|m)$-dimensional Lie superalgebra

$$\dim C^k = p\sum_{i=0}^{k}\binom{n}{k-i}\binom{m+i-1}{i} \equiv p\binom{n}{k} + p\sum_{i=1}^{k}\binom{n}{k-i}\binom{m+i-1}{i}.$$

In many cases it is possibly to extract some easier to handle subcomplexes of the full cochain complex (1). The partition of cochain complex for a graded algebra and module into homogeneous components is a typical example. In many papers (see, e. g., [10–12]) more special subcomplexes were used successfully to obtain new results in the theory of cohomology of Lie (super)algebras.[2]

The main idea of the new algorithm presented in [6–8] is to extract the minimal possible subcomplexes from complex (1) and to carry out computations within these subcomplexes. There are two versions of the algorithm. One of them is applied when the cochain spaces under consideration are infinite-dimensional (or their dimensions are too large to fit the available memory), but the minimal subcomplexes contain finite-dimensional spaces of k-cochains. Another version of the algorithm is applied when it is possible to construct the full space C^k. Below we present this version in the pseudocode form.
Here the subalgorithm **GenerateMonomials** generates the full set M_g^k of super skew-symmetric monomials

$$c(\alpha_{i_1}, \ldots, \alpha_{i_k}; \xi_\iota) \equiv c(\alpha_{i_1}) \wedge \cdots \wedge c(\alpha_{i_k}) \otimes \xi_\iota \equiv \alpha'_{i_1} \wedge \cdots \wedge \alpha'_{i_k} \otimes \xi_\iota$$

forming basis of the cochain space C^k in the grade g; $\alpha_{i_j} \in A$ and $\xi_\iota \in X$ are basis elements of algebra and module; α'_i is the dual to α_i element. The subalgorithm **ChooseMonomial** takes some monomial $m_g^k \in M_g^k$. This monomial is a starting monomial for constructing the subcomplex s by the subalgorithm **ConstructSubcomplex**. The subalgorithm **ComputeCohomologyInSubcomplex** computes basis cohomological classes $BH_{g,s}^k$ in the subcomplex s by the straightforward algorithm described above.

2 Computation of $H_g^k(\mathrm{H}(2|0))$

In this section we present the results of computation of cohomology in the trivial module for Lie algebra H(2|0) of formal hamiltonian vector fields on the 2-dimensional symplectic manifold. We describe also

[2] The main trick consists in imposing some restrictions on the elements of C^k and proving the invariance of these restrictions with respect to the differential.

Algoritm: ComputeCohomology

Input: A, Lie (super) algebra; X, module;
k, cohomology degree; g, grade
Output: BH_g^k, set of basis cohomological classes
Local: M_g^k, full set of k-cochain monomials (basis of C_g^k);
s, current subcomplex: $C_{g,s}^{k-1} \xrightarrow{d_{g,s}^{k-1}} C_{g,s}^k \xrightarrow{d_{g,s}^k} C_{g,s}^{k+1}$;
$m_g^k \in M_g^k$, starting monomial for constructing subcomplex s;
$M_{g,s}^k$, set of k-cochain monomials involved in subcomplex s;
$BH_{g,s}^k$, set of basis cohomological classes in subcomplex s

$BH_g^k := \emptyset$
$M_g^k :=$ **GenerateMonomials**(A, X, k, g)
while $M_g^k \neq \emptyset$ **do**
    $m_g^k :=$ **ChooseMonomial**(M_g^k)
    $\{s, M_{g,s}^k\} :=$ **ConstructSubcomplex**(m_g^k)
    $BH_{g,s}^k :=$ **ComputeCohomologyInSubcomplex**(s)
    if $BH_{g,s}^k \neq \emptyset$ **then**
        $BH_g^k := BH_g^k \cup BH_{g,s}^k$
    fi
    $M_g^k := M_g^k \setminus M_{g,s}^k$
od
return BH_g^k

the cohomological classes up to grade 8 for the Poisson algebra Po(2|0) which is a central extension of the algebra H(2|0).

The hamiltonian algebra H(2n|m) is an algebra of vector fields (see, e.g., [9]) acting on the $(2n|m)$ supermanifold and preserving the following 2-form

$$\sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{j=1}^{m} d\theta_j \wedge d\theta_j,$$

where $p_1, \ldots, p_n; q_1, \ldots, q_n$ and $\theta_1, \ldots \theta_m$ are even and odd local variables on the supermanifold, respectively. The elements of H(2n|m) can be expressed in terms of *generating function* $f(p_1, \ldots, p_n; q_1, \ldots, q_n;$ $\theta_1, \ldots \theta_m)$ by the formula

$$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j=1}^{m} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j}, \tag{6}$$

where $p(f)$ is parity of the function f (this function is called usually *hamiltonian*). Thus one can consider the formal hamiltonian vector fields as linear combinations of monomials in the variables p_i, q_i and θ_j (except for the monomial 1). Considering these monomials as basis elements of H(2n|m) and using prescribed $\mathbb{Z}$-grading for the variables p_i, q_i and θ_j one can impose $\mathbb{Z}$-grading gr() on the algebra H(2n|m). The standard grading is $\mathrm{gr}(p_i) = \mathrm{gr}(q_i) = \mathrm{gr}(\theta_j) = 1$. For the standard grading the grade of algebra element corresponding to some monomial is equal to the grade of this monomial minus 2 (due to two differentiations in the terms of expression (6)).

Since the hamiltonian algebra is very important in both classical and quantum physics many efforts were applied to the investigation of its cohomology. Most advanced results were obtained for the finite-dimensional algebras of the form H(0|m) [13, 14]. Nevertheless the hamiltonian algebras on supermanifolds with nonzero even dimension are more important in applications but computation of their cohomologies is a much more difficult task. Some results about cohomologies of such algebras were obtained in [10–12]. In the paper [10] some elements of $H_g^k(\mathrm{H}(2|0))$ were calculated by considering special subcomplexes (and using computer partially). We present here all cohomological classes (without discussing their meaning and interpretation) from $H_g^k(\mathrm{H}(2|0))$ up to grade 8.

The computed results are summarized in Table 1. The boxes of this table corresponding to cochain degree k and cochain grade g contain the following information: $\dim C_g^k$, dimension of the full space of k-cochains in grade g; number of minimal subcomplexes $C_{g,s}^{k-1} \xrightarrow{d_{g,s}^{k-1}} C_{g,s}^k \xrightarrow{d_{g,s}^k} C_{g,s}^{k+1}$ extracted by the algorithm from the full complex; $\max\dim C^k_{g,s}$, maximum dimension of the subspace of (k,g)-cochains among all subcomplexes. The empty box means that $\dim C^k_g = 0$, i.e. the space of (k,g)-cochains is empty. The boxes marked by the bullet • contain nontrivial 1-dimensional cohomological classes. For example,

Table 1. Computation of H^k_g for $(k,g) \in [1,\ldots,\infty) \otimes [-2,\ldots,8]$

$k\backslash g$	-2	-1	0	1	2	3	4	5	6	7	8
1		2 2 1	3 3 1	4 4 1	5 5 1	6 6 1	7 7 1	8 8 1	9 9 1	10 10 1	11 11 1
2	1 1 • 1	6 4 2	11 5 3	22 6 5	33 7 7	52 8 9	71 9 11	100 10 14	129 11 17	170 12 20	211 13 23
3	3 3 1	10 4 3	30 7 8	60 8 13	116 9 22	200 10 34	326 11 52	502 12 72	752 13 100	1078 14 133	1515 15 177
4	3 3 1	14 6 4	45 7 11	108 8 22	242 11 44	466 12 78	857 13 135	1468 14 210	2426 15 326	3820 16 478	5872 17 698
5	1 1 • 1	12 6 3	41 7 9	128 10 25	315 11 59	706 12 117	1432 13 222	2748 16 391	4949 17 671	8568 18 1078	14240 19 1710
6		4 4 1	23 7 5	90 10 18	264 11 50	688 12 114	1580 15 246	3382 16 483	6734 17 916	12766 18 1619	23074 19 2806
7			6 5 • 2	32 8 7	135 11 25	412 12 70	1128 15 180	2730 16 396	6132 17 842	12818 18 1649	25488 21 • 3148
8				4 4 1	33 9 7	138 10 25	479 13 79	1388 16 207	3606 17 510	8546 18 1125	18963 21 2391
9					1 1 1	20 8 4	99 11 17	396 14 62	1260 15 188	3576 18 489	9104 19 1188
10							5 5 1	46 10 8	217 13 35	818 16 120	2578 17 • 358
11									10 7 2	70 10 12	350 15 54
12											10 7 2

the box corresponding to the pair $(k,g) = (7,8)$ tells that $\dim C^7_8 = 25488$, the number of subcomplexes is 21, $\max\dim C^7_{8,s} = 3148$ and $\dim H^7_8 = 1$. More detailed information about computation in $(k,g) = (7,8)$ is given in Table 2. In this table the columns $\dim Z^k_{g,s}$, $\dim B^k_{g,s}$ and $\dim H^k_{g,s}$ contain the dimensions of cocycle, coboundary and cohomology spaces in subcomplexes, respectively. On can see that there are 10 pairs of subcomplexes with repeated structure and the only single subcomplex containing nontrivial cohomological class.

The full set of nontrivial (1-dimensional) cohomological classes is: $H^2_{-2}, H^5_{-2}, H^7_0$ (computed earlier) and H^7_8, H^{10}_6 (computed by the new program). As to the Poisson algebra Po(2|0), the part of its cohomological classes up to grade 8 coincides with those for H(2|0) except for H^2_{-2}.[3] $H^k_{g\leq 8}(\mathrm{Po}(2|0))$ contains

[3] The cocycle H^2_{-2} describes the central extension of the algebra H(2|0) to Po(2|0).

Table 2. Subcomplex structure for $(k, g) = (7, 8)$

$\dim C^{k-1}_{g,s}$	$\dim C^{k}_{g,s}$	$\dim C^{k+1}_{g,s}$	$\dim Z^{k}_{g,s}$	$\dim B^{k}_{g,s}$	$\dim H^{k}_{g,s}$	repeated
0	1	1	0	0	0	2
12	17	11	9	9	0	2
72	80	54	43	43	0	2
223	243	167	130	130	0	2
507	540	375	292	292	0	2
909	976	702	520	520	0	2
1406	1536	1120	813	813	0	2
1928	2117	1578	1114	1114	0	2
2382	2652	1992	1387	1387	0	2
2695	3008	2286	1568	1568	0	2
2806	3148	2391	1640	1639	1	1

also four additional classes: $H^{6}_{-4}, H^{8}_{-2}, H^{8}_{6}, H^{11}_{6}$. But all these classes are multiplicative consequences of the classes $H^{5}_{-2}, H^{7}_{0}, H^{7}_{8}, H^{10}_{6}$. These classes can be expressed in the form $H^{k}_{g} = H^{k-1}_{g+2} \wedge c(\mathcal{Z})$ due to the general property [4] of cohomology of algebras containing central element $\mathcal{Z}$.

3 Conclusion

Our new algorithm demonstrates a substantially higher efficiency in comparison with the old one. For example, the program described in [2] computes the case $(k, g) = (6, 5)$ in 35 min 45 sec = 2145 sec whereas the new program takes 54 sec for this task. For both runs we used PC Pentium III, 667MHz, 256MB RAM. The superiority of the new program grows with increasing of the task complexity. Nevertheless, due to rapidly increasing computational complexity the presented results are not sufficient to derive any general idea about the structure of cohomology ring $H^{*}_{*}(\mathrm{H}(2|0))$. Our computation was carried out over the field of rational numbers $\mathbb{Q}$. As profiling shows, the most time consuming part of computation by the program **LieCohomology** is multiprecision arithmetic. This is common difficulty for almost all problems in computer algebra. It seems that carrying out computation over the finite fields, say $\mathbb{Z}_p$, we can go to the grade 40–50 for the problem considered here, but the results obtained in this way can be considered merely as hints.

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