Classification and Applications of Monomial Orderings and the Properties of Differential Orderings

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Abstract. We consider monomial orderings specified by matrices of a special form and suggest a new proof of the well-known fact that every monomial ordering can be obtained in this way. The relations between matrices specifying the same ordering are discussed and the *canonical* form of a monomial matrix is presented. We give some applications of this ordering presentation to Gröbner bases of ideals. We also discuss orderings on differential variables and differential monomials. We prove the property of well-ordering on differential monomials using only two source properties and without any additional conditions.

1 Monomial Orderings

1.1 Introduction

A set of *monomials* in n variables can be considered as the set of the formal expressions $\mathbf{M}^n = \{x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \mid i_1, i_2, \dots, i_n \in \mathbb{N}_0\}$ w.r.t. the usual multiplication (addition of exponents). That is the so called multiplicative form of a monomial. We shall, however, also use an additive form: denote $x = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ by the vector $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n) \in \mathbb{N}_0^n$. The product of monomials $x \cdot y$ corresponds to the sum of the vectors $\alpha + \beta$ and vice versa.

Let a total ordering $<$ on $\mathbf{M}^n = \mathbb{N}_0^n$ be fixed, i.e. any two different monomials are comparable and $<$ is irreflexible, asymmetric and transitive. We say that this ordering is a *monomial* ordering if the following conditions hold:

(I) $$0 < \alpha \ \forall \alpha \neq 0,\ \alpha \in \mathbb{N}_0^n,$$

(II) $$\alpha < \beta \Rightarrow \alpha + \gamma < \beta + \gamma \ \forall \alpha, \beta, \gamma \in \mathbb{N}_0^n.$$

One can prove that $\mathbb{N}_0^n$ is well ordered with respect to any total ordering satisfying the first of these conditions (see, for example, [2]).

Example 1. Lexicographic ordering (*lex*), total degree and then lexicographic ordering (*deglex*), total degree and then reverse lexicographic ordering (*degrevlex*). See [2] for details.

1.2 Existence of Representation

One can find the proofs of the following Theorems 1, 2 and 4 in [4], [5], [6]. As it is mentioned in [4], an abstract solution is proposed in [8] and a vectorial approach is presented in [3]. These results are well-known, but we present our own proofs using only elementary methods.

Theorem 1. *Let $A = (a_{ij})$ be an n by k matrix over $\mathbb{R}$ with the following two properties: the rank of the matrix is n and for each row of A, the first non-zero element in this row is positive. Then, the ordering on $\mathbb{N}_0^n$ such that*

$$\alpha < \beta \Leftrightarrow \alpha A <_{lex} \beta A$$

is monomial.

Proof. Let us check the properties of monomial orderings. Note that the second property of the matrix can be formulated in this way: *each row of the matrix A is lexicoghraphically greater than zero vector.* Using this fact, we get:

$$0A = 0 <_{lex} \beta A \; \forall \beta \in \mathbb{N}_0^n, \; \beta \neq 0,$$
$$\alpha A <_{lex} \beta A \Rightarrow (\alpha + \gamma)A = \alpha A + \gamma A <_{lex} \beta A + \gamma A = (\beta + \gamma)A.$$

Therefore, this ordering is monomial.

Our aim is to prove the converse result in an elementary way. Let $\mathbb{R}_+^n = [0, +\infty)^n \setminus \{0\}$ be the set of all vectors with non-negative coordinates without the zero vector. By uv we denote the standard scalar product of vectors $u, v \in \mathbb{R}^n$ and by v^j we denote the jth coordinate of the vector v. Coefficients will also be written with superscripts.

Theorem 2. *Consider a monomial ordering $<$ on $\mathbb{N}_0^n$. Then, there exists a vector $v \in \mathbb{R}_+^n$ such that*

$$\alpha < \beta \Rightarrow v\alpha \leqslant v\beta \; \forall \alpha, \beta \in \mathbb{N}_0^n. \tag{1}$$

Proof. The proof is by *reductio ad absurdum*. We shall show that there exists a finite set of monomial pairs (α, β), $\alpha < \beta$ such that each vector of $\mathbb{R}_+^n$ changes the ordering of at least one pair. Then we shall prove that under certain conditions there are exactly n such pairs. Using the properties of monomial orderings, we shall obtain a contradiction: $\alpha \geqslant \beta$.

Assume that the converse is true, i.e.

$$\forall v \in \mathbb{R}_+^n \; \exists \alpha, \beta \in \mathbb{N}_0^n, \; \alpha < \beta, \; v\alpha > v\beta.$$

Consider the sets $V_{(\alpha,\beta)} = \{v \in \mathbb{R}^n \mid v\alpha > v\beta\}$, $\alpha, \beta \in \mathbb{N}_0^n$, $\alpha < \beta$.

These sets are obviously open in $\mathbb{R}^n$, because they form the semispaces without a hyperplane. However,

$$V = \bigcup_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ \alpha < \beta}} V_{(\alpha,\beta)} \supset \mathbb{R}_+^n.$$

Let S_+ be the non-negative part of the sphere S^{n-1} in $\mathbb{R}^n$: $S_+ = S^{n-1} \cap \mathbb{R}_+^n$. This set is obviously a compact set in $\mathbb{R}^n$ and $S_+ \subset V$. Hence, we get a finite number of the sets $V_i = V_{(\alpha_i,\beta_i)} \cap \mathbb{R}_+^n$, $i = \overline{1,m}$ such that $S_+ \subset \bigcup_{i=1}^{m} V_i$.

Consider a vector $v \in \mathbb{R}_+^n$. There exist $\alpha, \beta \in \mathbb{N}_0^n$, $\alpha < \beta$ such that $v \in V_{(\alpha,\beta)}$. It is clear that $cv \in V_{(\alpha,\beta)}$ $c \in \mathbb{R}$, $c > 0$. It follows that $v \in V_{(\alpha,\beta)} \Leftrightarrow \frac{v}{\|v\|} \in V_{(\alpha,\beta)}$. However, $\frac{v}{\|v\|} \in S_+$ implies that $\exists i \in \overline{1,m} : \frac{v}{\|v\|} \in V_i$. Hence,

$$\forall v \in \mathbb{R}_+^n \; \exists i \in \overline{1,m} : v \in V_i. \tag{2}$$

For all $\gamma \in \mathbb{N}_0^n$ we have $V_{(\alpha,\beta)} = V_{(\alpha+\gamma,\beta+\gamma)}$, because $v\alpha > v\beta \Leftrightarrow v(\alpha + \gamma) > v(\beta + \gamma)$. Thus, we can assume that the second components of the pairs, β_i, are the same. In fact, let us change $V_i = V_{(\alpha_i,\beta_i)}$ for

$$V_{(\alpha_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} \beta_j, \sum_{j=1}^{m} \beta_j)}.$$

We denote the second (common) component of the pairs by β.

Consider the set V_j such that $V_j \subset \bigcup_{\substack{i=1 \\ i \neq j}}^{m} V_i$. Then, in (2), we can omit this set. Hence, we may assume that for all sets V_j there exists a vector $v_j \in \overline{V_j} := V_j \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{m} V_i \neq \varnothing$. Clearly, $\overline{V_i} \cap \overline{V_j} = \varnothing$, $i \neq j$. The vectors v_i have the following properties:

$$v_i\alpha_i > v_i\beta \tag{3}$$
$$v_i\alpha_j \leqslant v_i\beta \quad \forall j = \overline{1,m}, \; j \neq i. \tag{4}$$

We claim that the vectors $v_i,\ i=\overline{1,m}$ are linearly independent. Indeed, suppose that there is a non-trivial linear combination

$$a^1v_1+a^2v_2+\ldots+a^mv_m=0,\ a^i\in\mathbb{R},\ i=\overline{1,m}.$$

Since the components of the vectors v_i are non-negative and these vectors themselves are not equal to zero, all coefficients a^i cannot have the same sign. Moving negative summands to the right-hand side and, possibly, renumbering vectors and coefficients, we obtain:

$$v:=c^1v_1+\ldots+c^kv_k=d^{k+1}v_{k+1}+\ldots+d^mv_m, \tag{5}$$
$$v\neq 0,\ c^1,\ldots,c^k,d^{k+1},\ldots,d^m\geqslant 0,$$

and $v\in\mathbb{R}^n_+$. By assumption, there is a pair $(\alpha_i,\beta),\ \alpha_i<\beta$ such that $v\alpha_i>v\beta$. However, if $1\leqslant i\leqslant k$, then

$$v\alpha_i=(d^{k+1}v_{k+1}+\ldots+d^mv_m)\alpha_i=d^{k+1}v_{k+1}\alpha_i+\ldots+d^mv_m\alpha_i\leqslant$$
$$\leqslant d^{k+1}v_{k+1}\beta+\ldots+d^mv_m\beta=v\beta,$$

since (4) holds. Similarly, if $k+1\leqslant i\leqslant m$, then

$$v\alpha_i=(c^1v_1+\ldots+c^kv_k)\alpha_i\leqslant(c^1v_1+\ldots+c^kv_k)\beta=v\beta.$$

Thus, we have a contradiction: $v\alpha_i\leqslant v\beta\ \forall i=\overline{1,m}$. This means that the vectors v_i are linearly independent and, in particular, the number of pairs $m\leqslant n$. Let us show that $m=n$.

Introduce the notation

$$T_i=\{\alpha\in\mathbb{N}^n_0,\ \alpha<\beta\mid\forall v\in V_i\ v\alpha>v\beta\}\subset\mathbb{N}^n_0.$$

Since $<$ is a well order on $\mathbb{N}^n_0$, there exists a minimal element in any non-empty subset of $\mathbb{N}^n_0$. Suppose $\tilde\alpha_i=\min\limits_{\alpha\in T_i}\alpha$. By construction, we have $V_{(\alpha_i,\beta)}\subset V_{(\tilde\alpha_i,\beta)}$. Therefore, as in (2), we have

$$\forall v\in\mathbb{R}^n_+\ \exists i\in\overline{1,m}:\ v\in\tilde V_i:=V_{(\tilde\alpha_i,\beta)} \tag{6}$$

Now let $\gamma\in\mathbb{N}^n_0$ be an arbitrary non-zero monomial. From the first property of monomial orderings, it follows that

$$\gamma\mid\alpha\iff\exists\delta\in\mathbb{N}^n_0:\gamma+\delta=\alpha\implies\gamma\leqslant\alpha.$$

In particular, if $\gamma\mid\tilde\alpha_i,\ v_i\in V_i$, then $\tilde\alpha_i-\gamma<\tilde\alpha_i$, and, hence, we obtain

$$\tilde\alpha_i-\gamma\notin T_i\implies v_i(\tilde\alpha_i-\gamma)\leqslant v_i\beta\iff v_i\tilde\alpha_i\leqslant v_i\gamma+v_i\beta.$$

Thus, since $v_i\tilde\alpha_i>v_i\beta$, we have

$$0<v_i\tilde\alpha_i-v_i\beta\leqslant v_i\gamma. \tag{7}$$

Now consider the standard base vectors $e_j\in\mathbb{R}^n_+$ (all components except for the jth one are zeroes and the jth component equals one). By assumption, for each such vector, there exists a number $i=i(j)$: $e_j\in\tilde V_i$. Suppose that there exists at least one non-zero component $\tilde\alpha^k_i\neq 0,\ k\neq j$, of $\tilde\alpha_i$. Formally, we set $\gamma=\tilde\alpha^k_ie_k$; i.e., γ is the projection of $\tilde\alpha_i$ onto e_k. By construction, $\gamma|\tilde\alpha_i$. Combining this with (7), we obtain a contradiction:

$$0<e_j\tilde\alpha_i-e_j\beta\leqslant e_j\gamma=\tilde\alpha^k_ie_je_k=0\ (j\neq k)\Rightarrow 0<0.$$

This implies that the monomial $\tilde\alpha_i,\ i=i(j)$, which corresponds to the vector e_j, can be written as

$$\tilde\alpha_i=a^je_j, \tag{8}$$

where $a^j\in\mathbb{N}$ is a coefficient. This means that the number of pairs of monomials is $m\geqslant n$, because a^je_j and a^ie_i are different monomials, $i\neq j$. On the other hand, by the aforesaid, $m\leqslant n$. Thus, we have $m=n$ and all monomials $\tilde\alpha_i$ have the form as in (8).

For simplicity, in what follows, we write α_i instead of $\tilde\alpha_i$. Let us embed the monomials α_i in $\mathbb{R}^n$ and construct a hyperplane Γ which contains these monomials. It is clear that the vector $l=(\frac{1}{a^1},\ldots,\frac{1}{a^n})\in\mathbb{R}^n_+$ is the normal vector to Γ, since $l\alpha_i=1,\ i=\overline{1,m=n}$. If we have $l\beta\geqslant 1$, then we get the contradiction immediately: there is no pair $(\tilde{\alpha}_i, \beta)$ for l to change the ordering. This yields $l\beta < 1$. We may represent β as a linear combination:

$$\beta = \sum_{i=1}^{n} c^i \alpha_i,\ c^i = \frac{\beta^i}{a^i} \in \mathbb{Q},\ c^i \geqslant 0$$

(this is possible, since the vectors α_i form a basis of $\mathbb{R}^n$). In these terms,

$$C := l\beta = \sum_{i=1}^{n} c^i l\alpha_i = \sum_{i=1}^{n} c^i < 1.$$

We now set

$$p^i := \frac{Mc^i}{1-C} \geqslant 0,$$

where M is a sufficiently large natural number such that all $p^i \in \mathbb{N}_0$.

Consider the formal monomial

$$\overline{\beta} := \beta + \sum_{i=1}^{n} p^i(\beta - \alpha_i).$$

Some components of this vector may be negative. To make it meaningful, we may assume that we have shifted the monomials α_i and β along some large vector (monomial) δ. This shifting takes the vector $\overline{\beta}$ to the vector $\overline{\beta} + \delta$, making its components positive integers. Note that our proof does not depend on δ. Consequently we don't do that. When we write $\overline{\beta} < \beta$, we mean that $\overline{\beta} + \delta < \beta + \delta$. We get:

$$\begin{aligned}\overline{\beta} &= \sum_{i=1}^{n} c^i\alpha_i + \sum_{i=1}^{n} p^i(\sum_{j=1}^{n} c^j\alpha_j - \alpha_i) \\ &= \sum_{i=1}^{n} c^i\alpha_i + \sum_{j=1}^{n}(\sum_{i=1}^{n} p^i)c^j\alpha_j - \sum_{i=1}^{n} p^i\alpha_i = \sum_{i=1}^{n}(c^i + c^i\sum_{j=1}^{n} p^j - p^i)\alpha_i.\end{aligned}$$

Let us look at the coefficients of this decomposition. We may write:

$$\begin{aligned} c^i + c^i\sum_{j=1}^{n} p^j - p^i &= c^i\Big(1 + \frac{M}{1-C}\sum_{j=1}^{n} c^j\Big) - \frac{Mc^i}{1-C} \\ &= c^i\Big(1 + \frac{MC}{1-C} - \frac{M}{1-C}\Big) = \frac{c^i}{1-C}(1 - C + MC - M) = c^i(1-M) \leq 0,\end{aligned}$$

since $M \geqslant 1,\ c_i \geqslant 0$. By the previous formula we obtain the chain of monomial inequalities:

$$\overline{\beta} \mid 0 \Rightarrow \overline{\beta} \leqslant 0 < \alpha_i < \beta\ \forall i = \overline{1,n}.$$

At the same time, since $\beta - \alpha_i > 0$, we have

$$\overline{\beta} = \beta + \sum_{i=1}^{n} p^i(\beta - \alpha_i) > \beta.$$

Comparing this result with the previous one, we obtain by transitivity that $\beta < \beta$. But a monomial ordering must be irreflexible. This contradiction concludes the proof.

Theorem 3. *Let $K \subset \mathbb{N}_0^n$ be a closed w.r.t. addition subset of monomials in n variables. Consider a total ordering $<_K$ on K satisfying the following conditions:*

$$\alpha \leqslant_K \alpha + \gamma\ \ \forall \alpha \in K,\ \gamma \in \mathbb{N}_0^n \text{ such that } \alpha + \gamma \in K, \tag{9}$$

$$\alpha <_K \beta \Rightarrow \alpha + \gamma <_K \beta + \gamma \tag{10}$$

$$\forall \alpha, \beta \in K,\ \gamma \in \mathbb{Z}^n \text{ such that } \alpha + \gamma \in K,\ \beta + \gamma \in K.$$

Then $<_K$ can be extended to the monomial ordering $<$ on $\mathbb{N}_0^n$ such that

$$\alpha <_K \beta \iff \alpha < \beta\ \forall \alpha, \beta \in K.$$

Proof. Consider the set of all non-ordered pairs of different monomials

$$P = \{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbb{N}_0^n,\ \alpha \neq \beta\}.$$

This set is obviously countable. Let us number its elements. We must learn to compare the elements of any pair in P and that comparison should be well-defined.

Suppose $V_1 = \{\beta - \alpha \in \mathbb{Z}^n \mid \alpha, \beta \in K,\ \alpha <_K \beta\} \cup \mathbb{N}_0^n \setminus \{0\}$. Consider the set of monomials

$$P_1 = \{\{\gamma, \gamma + v\} \mid \gamma \in \mathbb{N}_0^n,\ v \in V_1,\ \gamma + v \in \mathbb{N}_0^n\} \subset P.$$

Let us introduce the ordering on these pairs. We say that $\gamma < \gamma + v$ if $p_1 \in P_1,\ p = \{\gamma, \gamma + v\}$. This definition is well-defined and agrees with the ordering $<_K$. Indeed, if $v \in V_1$ then $-v \notin V_1$, and thus the pair p cannot be represented as $p = \{\gamma, \gamma - v\}$. Therefore, for any such pair p, the vector $v = \beta - \alpha \in V_1$ is uniquely defined. If $\gamma,\ \gamma + v \in K$ then $\gamma + \alpha,\ \gamma + \beta \in K$ and $\gamma + \alpha <_K \gamma + \beta$. Hence $\gamma <_K \gamma + \beta - \alpha$ (property (10)). At the same time if $\gamma <_K \gamma + \beta - \alpha$ then $\alpha <_K \beta$ and $v = \beta - \alpha \in V_1$.

It is clear that $\gamma \leqslant \gamma + \delta\ \forall \gamma, \delta \in \mathbb{N}_0^n$ (if $\delta \neq 0$, then we may take $v = \delta \in V_1$). Besides that $\gamma_1 < \gamma_2 \Rightarrow \gamma_1 + \delta < \gamma_2 + \delta\ \forall \gamma_1, \gamma_2, \delta \in \mathbb{N}_0^n$. Actually, $v = (\gamma_2 + \delta) - (\gamma_1 + \delta) = \gamma_2 - \gamma_1 \in V_1$. We see that all properties of monomial orderings are satisfied.

Let the inequality $P_1 \neq P$ hold; then we have not extended the ordering to all pairs yet. Take the pair $p = \{\alpha, \beta\}$ with the minimal number in $P \setminus P_1$. Let us say that $\alpha < \beta \Leftrightarrow \alpha <_{lex} \beta$. Construct the sets $V_2 = V_1 \sqcup \{\beta - \alpha\}$ and $P_2 = P_1 \sqcup \{\{\gamma, \gamma + (\beta - \alpha)\}\}$. It is not hard to prove that for all pairs in P_2 the conditions as above hold.

If we continue this process, we either stop at some step, or get a sequence $P_1, P_2, P_3, \ldots$ with the condition $\bigcup\limits_{i=1}^{\infty} P_i = P$. We stress that for all pairs p of different monomials there exists a number m such that $p \in P_m$, and, therefore, we are able to compare the elements of this pair. The ordering specified in this way is monomial and agrees with $<_K$.

Theorem 4. *Consider a monomial ordering $<$ on the set of monomials in n variables. Then there exists a matrix $A = (a_{ij})$ over $\mathbb{R}$ of the size n by k and of the rank k such that*

$$\alpha < \beta \iff \alpha A <_{lex} \beta A, \tag{11}$$

and, moreover, the first non-zero element in each row of this matrix is positive.

Proof. We shall prove this theorem by induction on the number of variables. Since in the one-dimensional case there is only one monomial ordering (by degree), any matrix $A = (\lambda)$ $\lambda > 0$ of the size 1 by 1 satisfies the theorem. Suppose we can find such matrix for any monomial ordering in $n - 1$ variables, $n \geqslant 2$. Let us show that it is possible to construct a matrix for the case of n variables.

Take a vector $v \in \mathbb{R}_+^n$ such that $\alpha < \beta \Rightarrow v\alpha \leqslant v\beta\ \forall \alpha, \beta \in \mathbb{N}_0^n$. The existence of the vector v is guaranteed by Theorem 2. Consider the set of pairs of monomials which cannot be compared when using vector v:

$$E = \{(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \alpha < \beta,\ v\alpha = v\beta\}.$$

If this set is empty, then we have already constructed the matrix (it consists of one column). Otherwise, notice that monomials in each pair $(\alpha, \beta) \in E$ belong to a hyperplane (which, clearly, depends on pair). This hyperplane is orthogonal to v. Consider the family of such hyperplanes which contain at least one integer point in $\mathbb{N}_0^n$. By M_Γ denote the set of all monomials in the hyperplane Γ and by α_Γ denote the minimal element of the set M_Γ (it always exists, since our ordering is monomial).

Now suppose that there are only finitely many hyperplanes $\{\Gamma\}$ such that the element α_Γ belongs to some coordinate axis, i.e. $\alpha_\Gamma = (0, \ldots, 0, a_\Gamma^k = A_\Gamma, 0, \ldots, 0)$. Choose the maximal component $A = \max A_\Gamma$. We can find the hyperplane Φ in the family such that for all monomials in M_Φ we have that at least one of its components is greater than A. Let us evaluate the minimal monomial $\alpha_\Phi = (\alpha_\Phi^1, \ldots, \alpha_\Phi^n)$ in Φ. This monomial cannot belong to any coordinate axis. Moreover, $\alpha_\Phi^k > A$ for some k. Let us shift all monomials in M_Φ along the integer vector $(-\alpha_\Phi^1, \ldots, -\alpha_\Phi^{k-1}, 0, -\alpha_\Phi^{k+1}, \ldots, -\alpha_\Phi^n)$. (Some monomials may not remain non-negative; then we exclude them.) We claim that this shifting takes the hyperplane Φ to some other hyperplane Φ' such that the monomial α_Φ is in the kth coordinate axis, and it remains minimal in $M_{\Phi'}$ as before. This shifted monomial has the form $(0, \ldots, \alpha_\Phi^k, \ldots, 0)$, $\alpha_\Phi^k > A$. This contradicts A being maximal. Hence, there are infinitely many hyperplanes with the property as above. This means that there exists a coordinate axis x_m containing infinitely many monomials of the form α_Γ. Let us fix this value of m.

Consider the family H of all v-orthogonal hyperplanes in $\{\Gamma\}$ which intersect the x_m axis in some integer point β_Γ. We claim that this point is the minimal monomial in Γ, i.e. $\beta_\Gamma = \alpha_\Gamma$. Indeed, for any hyperplane $\Gamma \in H$ there exists a hyperplane Φ such that $\alpha_\Phi^m > \beta_\Gamma^m$ by the above. Shifting along the vector $\alpha_\Phi - \beta_\Gamma$, we obtain that β_Γ is the minimal element in M_Γ as well.

Now consider a projection along x_m:

$$\phi : H \cap \mathbb{N}_0^n \to \mathbb{N}_0^{n-1}.$$

Let α be in the hyperplane $\Gamma \in H$. Consider

$$\alpha = (\alpha^1, \ldots, \alpha^n) \overset{\phi}{\mapsto} \alpha' = (\alpha^1, \ldots, \alpha^{m-1}, \alpha^{m+1}, \ldots, \alpha^n).$$

Denote $K = \mathrm{Im}(\phi) \subset \mathbb{N}_0^{n-1}$. Now we can construct the ordering $<_K$ on K. Let $\alpha', \beta' \in K$, $\alpha \in \phi^{-1}(\alpha')$, $\beta \in \phi^{-1}(\beta')$ and let α and β be in the same hyperplane (by construction of H, it is possible to do so). We say that $\alpha' <_K \beta'$ if $\alpha < \beta$. This ordering is well-defined, since different elements of the pre-image $\phi^{-1}(\alpha')$ differ only in the mth component. Further, if (α_1, β_1) and (α_2, β_2) are different pairs of the pre-images (α', β') and the elements of each pair are in the same hyperplane in H, then we obtain $(\alpha_2, \beta_2) = (\alpha_1 + \gamma, \beta_1 + \gamma)$ (or vice versa). By the second property of monomial orderings we get

$$\alpha_1 < \beta_1 \Leftrightarrow \alpha_2 < \beta_2 \Rightarrow \alpha' <_K \beta' \text{ is well-defined.}$$

It is easy to prove that $<_K$ satisfies the conditions of Theorem 3. Indeed, we have that any element in $\phi^{-1}(0)$ is minimal in its hyperplane. This means that $0 \leqslant_K \alpha' \; \forall \alpha' \in K$. Furthermore, let $\alpha' <_K \beta'$, $\gamma' \in \mathbb{Z}_0^{n-1}$ and $\alpha', \beta', \alpha' + \gamma', \beta' + \gamma' \in K$. In this case for $\alpha' + \gamma'$ and $\beta' + \gamma'$ we can find two pre-images of the form $\alpha + \delta, \beta + \delta$, $\delta \in \mathbb{Z}_0^{n-1}$. Evidently, $\alpha + \delta < \beta + \delta \Rightarrow \alpha' + \gamma' <_K \beta' + \gamma'$. Besides, if $\alpha', \beta' \in K$, then $\alpha' + \beta'$ is also in K, i.e. K is closed w.r.t. addition.

Now, let us use Theorem 3. Let us extend the ordering $<_K$ to a monomial ordering $<_{n-1}$ on $\mathbb{N}_0^{n-1}$. By the inductive hypothesis, there exists a matrix B of the size $n-1$ by s specifying $<_{n-1}$. Inserting zeros as the kth row and the vector v as the first column, we obtain the matrix A of the size n by $s+1$. This matrix specifies the ordering $<$.

In fact, we can compare monomials of different v-orthogonal hyperplanes at the first step. Let α and β belong to the same v-orthogonal hyperplane. It is possible to find a hyperplane $\Gamma \in H$ which is situated above, and a monomial $\gamma \in \Gamma$ such that $\alpha \mid \gamma$. Let $\delta = \beta + (\gamma - \alpha) \in M_\Gamma$. We get

$$\alpha < \beta \Leftrightarrow \gamma < \delta \Leftrightarrow \gamma' <_K \delta' \Leftrightarrow \gamma' <_{n-1} \delta' \Leftrightarrow$$
$$\Leftrightarrow \gamma' B <_{lex} \delta' B \Rightarrow \gamma A <_{lex} \delta A \Leftrightarrow \alpha A <_{lex} \beta A.$$

On the other hand, if $\alpha > \beta$, then $\beta < \alpha$ and $\beta A <_{lex} \alpha A$. Therefore, the matrix A specifies the monomial ordering $<$. This completes the proof.

Remark 1. The matrix from this theorem is of the size n by m where $m \leqslant n$. Besides that, all its elements are non-negative.

Remark 2. The condition that the first non-zero element in a matrix is greater than zero is necessary. Let i be a number of the line with the opposite property; then $x_i > x_i^2$ in the sense of this ordering. Contradiction.

1.3 Classification of Monomial Orderings

Let us investigate the properties of the specification of monomial ordering described above. We shall present the admissible transformations of the monomial matrices and give a classification of them. First we consider a general case. We propose an independent proof of the fact presented in [5]. This classification is covered by Theorem 6.

We also consider a special class of rational orderings. The equivalence in this case (in the case of rational matrices) is given by Theorem 7. First of all, let us consider the lexicographic ordering. From now we shall use a multiplicative form of monomials.

Definition 1. *Two matrices C_1 and C_2 are said to be equivalent w.r.t. the monomial ordering $<$ iff they specify the same order as the above.*

It is absolutely clear that the identity matrix specifies the lexicographic ordering $x_1 > x_2 > \ldots > x_n$. Let us denote this ordering by $<_{lex}$. The following Lemma 1 and Lemma 2 describe all matrices which specify $<_{lex}$.

Lemma 1. *Each upper-triangle matrix with positive elements in the main diagonal gives us $<_{lex}$. Let us denote such matrices by U.*

Proof. Let $C = (c_{ij})$. Consider arbitrary monomials $\alpha = (\alpha^1, \ldots, \alpha^n)$ and $\beta = (\beta^1, \ldots, \beta^n)$. Let us denote by $<_{lexC}$ the order specified by the matrix C. Then $\alpha <_{lexC} \beta \iff \alpha C <_{lex} \beta C$. It is equivalent to

$$(\alpha^1 c_{11}, \alpha^1 c_{12} + \alpha^2 c_{22}, \ldots, \alpha^1 c_{1n} + \ldots + \alpha^n c_{nn}) <_{lex}$$
$$<_{lex} (\beta^1 c_{11}, \beta^1 c_{12} + \beta^2 c_{22}, \ldots, \beta^1 c_{1n} + \ldots + \beta^n c_{nn}).$$

If $\alpha^1 < \beta^1$, then $\alpha <_{lex} \beta$. Let $\alpha^1 = \beta^1$. We have

$$\alpha^1 c_{12} + \alpha^2 c_{22} < \beta^1 c_{12} + \beta^2 c_{22} \Longleftrightarrow \alpha^2 c_{22} < \beta^2 c_{22} \Longleftrightarrow \alpha^2 < \beta^2.$$

Continuing this process, we conclude that $\alpha <_{lexC} \beta$ iff vector $(\beta^1 - \alpha^1, \ldots, \beta^n - \alpha^n)$ has the first non-zero coordinate being positive. This is equivalent to $\alpha <_{lex} \beta$.

Lemma 2. *We can obtain $<_{lex}$ only by using matrices of the form described in Lemma 1. Multiplying any lex-matrix by the matrix of the same type as U, we obtain a lex-matrix again.*

Proof. Let us prove the first part of the lemma. Suppose we obtain $<_{lex}$ by the matrix $C = (c_{ij})_{i,j=1}^n$. Assume $c_{ij} \neq 0,\ i < j\ (*)$. Consider the element c_{ij} such that j is minimum with the property $(*)$. Then $c_{ij} > 0$, since this is the first non-zero element in the ith row. Then there exists $k \in \mathbb{N}$ such that $kc_{ij} < c_{jj}$. Because of the minimal property of j, we have $x_j^k <_{lexC} x_i$, but $i < j$. Contradiction. Now, let $c_{ii} \leqslant 0$ for some i. We have already proved that $c_{ij} = 0$ if $i < j$. Then $c_{ii} = 0$, otherwise $c_{ii} < 0$ is the first non-zero element in the row. Contradiction again. Consider c_{ii} with this property and maximal i. Then $i < n$ (if $c_{nn} = 0$, then x_n and 1 are incomparable elements in the sense of $<_{lex}$). Thus, we conclude that $c_{ii+1} \geqslant 0,\ c_{i+1\,i+1} > 0$. Therefore, $x_{i+1}^l <_{lex} x_i$ for some $l \in \mathbb{N}$. Contradiction.

The second part of the proof uses the following fact: if we multiply two U-matrices, we obtain an U-matrix again (it is easy to check).

We shall describe some matrices which do not change monomial orderings in the following sense:

Lemma 3. *Let U be a matrix such as in Lemma 1 and C specifies a monomial ordering $<$. Then the matrix CU specifies the same ordering $<$.*

Proof.

$$\alpha < \beta \iff \alpha C <_{lex} \beta C \iff (\alpha C)U <_{lex} (\beta C)U.$$

In these implications the matrix U is applied to the vector rather than to the monomial. This fact does not contradict anything (lexicographic ordering on $\mathbb{R}^n$ does not have the property of a well-order only). The lemma is proved.

Let us note that we can multiply the columns of monomial matrices by positive numbers and add linear combinations of previous columns to a column. Indeed, a multiplication of a monomial matrix by an U-matrix reflects all these transformations.

Definition 2. *We shall say that a column of a monomial matrix is main iff it contains the first non-zero element in some row. We shall call this element the main element in these column and row.*

Notice that a column may have more than one main element.

Theorem 5. *Each monomial ordering can be specified by a matrix with non-zero determinant and non-negative elements.*

Proof. Consider a monomial ordering $<$. By Theorem 4, there exists an $n \times m$ matrix C specifying $<$ such that all of its first non-zero elements in the rows are positive. By Lemma 3, we can multiply the matrix C from the right by U-matrices. Let us choose the first non-zero column in the matrix C.

Note that by Theorem 4 this column does not contain negative numbers. We have constructed a linear system of columns independent on $\mathbb{R}$. (It consists only of one non-zero column.) We shall follow this tactics in the next steps of the algorithm.

Let us use the elementary transformations of the columns of matrix C. We have described above some transformations that do not change the ordering. Let us walk through the matrix C column by column from the left to the right. We shall add new columns, linearly independent with the previous ones, from the matrix C to the system.

Suppose that at some step of the algorithm the column is linearly dependent with the system already constructed: $(v_{k+1} = a^1 v_1 + \ldots + a^k v_k)$. Then, making elementary transformations of the matrix C (subtracting a linear combination of the first k columns from the $(k+1)$th column), we obtain the matrix $C_1 = CU_1U_2 \ldots U_k$ (U_i is an upper-triangle with 1 at the diagonal, $i = 1, \ldots, k$), where the $(k+1)$th column is zero. Thus, we can exclude it from the matrix C_1.

Continuing this process we obtain the matrix C_l of the rank l which consists of l columns.

Let us consider a linearly independent system of columns $\{v_1, \ldots, v_l\}$. Suppose this system is the result of the algorithm described above. Let us extend it in arbitrary way to a basis of linear space $\mathbb{R}^n$. We shall insert vectors obtained by such operation at the end of the matrix C_l (from the right-hand side). We shall get the matrix C', $\det(C') \neq 0$.

Let us prove the second part of the theorem. Consider the first column of the matrix C' (it is non-negative). Multiplying it by sufficiently large positive number, we may add it to other columns. This transformation does not change the ordering according to the lemma above. By continuing we shall obtain a matrix with non-negative elements. The theorem is completely proved.

Example 2. Consider the monomial matrix

$$C = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

According to the algorithm, consider the first column. It forms a system of columns C_1. The second column is linearly dependent with the first. We can subtract the first column, multiplied by 2, from the second one. We obtain:

$$C_2 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let us delete the second column from the system C_2 (we obtain the matrix C_3) and add the second column of C_3 to the third one. The matrix $\mathbf{C}$ is the output of the algorithm:

$$C_3 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus, if we speak about a monomial matrix, we can assume that it satisfies the assertion of the previous theorem. Now we shall investigate monomial matrices from another point of view. We shall classify monomial matrices and provide the canonical form.

Corollary 1. *We can assume that the columns of a monomial matrix are orthogonal.*

Proof. The process of orthogonalisation of the finite number of vectors can be formulated in the language of applying U-matrices. First, we work with the first column of the matrix, then apply it to the second, etc.

Definition 3. *The monomial matrix is a matrix specifying a monomial ordering.*

First of all, let us transform the monomial matrix. We shall exclude non-necessary columns from this matrix. Let the new matrix consist of the minimal number of columns of the source matrix and specify the same ordering.

Remark 3. We shall build the system of minimal number of rows of the monomial matrix moving from the left column to the right. We shall delete columns which do not differ new monomials.

Lemma 4. *Let A and B be monomial matrices which specify a monomial ordering $<$ and satisfy all previous conditions (they are orthogonal, of the length 1, etc.). Then the first columns of A and B are equal.*

Proof. This fact is easy to understand if we consider the 2 or 3-dimensional case and draw the pictures. But we give the analytical proof.

Let a and b be the first columns of matrices A and B and $\alpha \in \mathbb{Q}^n$. A and B specify the same ordering. Then $(\alpha, a)(\alpha, b) \geqslant 0$. If $a = b$, then it is trivial. The idea of the proof is based on the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$. If $a \neq b$, we can choose an a-orthogonal vector and obtain the contradiction with $(\alpha, a)(\alpha, b) \geqslant 0$.

More formally: let a' be orthogonal to a, then the hyperplane $L : \{a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0\}$, $a = (a_1, a_2, \ldots, a_n)$, which is orthogonal to a, contains a'. Consider a sequence $\{\alpha_n\}, \alpha_i \in \mathbb{Q}^n, i \in \mathbb{N}$ with the properties $(\alpha_{2k}, a) > 0$, $(\alpha_{2k-1}, a) < 0$, $k \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = a'$. Using the inequality $(\alpha, a)(\alpha, b) \geqslant 0$, we obtain that $(\alpha_{2k}, b) \geqslant 0$, $(\alpha_{2k-1}, b) \leqslant 0$, $k \in \mathbb{N}$. Due to the fact that a scalar product is a continuous function we conclude that $(\lim_{n\to\infty} \alpha_n = a', b) = 0$. Since a and b are of the length 1 and $(a', b) = 0$, we conclude that $a = b$ or $a = -b$. The latter case cannot appear, since a and b form the first columns in the corresponding monomial matrices.

Theorem 6. *Let A and B be monomial matrices specifying the monomial ordering $<$. Then we can transform A to B and present the canonical matrix of $<$. This canonical matrix is uniquely determined.*

Proof. First of all, let us apply Lemma 4 to A and B. We obtain that the first columns of A and B are the same. At the next steps of comparing columns we may have problems.

At the first step we used the density of rational vectors. The collision is a consequence of the following fact. The dimension over $\mathbb{Q}$ of all rational vectors which go to zero after the first step of the comparison decreases rapidly. By this reason these vectors need to be compared by other columns. Let us note that the corresponding dimension over $\mathbb{R}$ decreases only by 1 as a dimension of solutions of one non-zero linear equation. We shall illustrate these facts in the example below.

By construction, the second column in our matrices is orthogonal to the first. Consider linear space L over $\mathbb{Q}$ generated by linearly independent rational vectors, which become zeros after the first step (i.e. which are orthogonal to the first column). Let us denote the corresponding linear space over $\mathbb{R}$ by M.

Let $k = \dim M - \dim L \geqslant 0$. If k is equal to zero, then we can apply the previous lemma, since the property of density holds in this case. Consider the case $k > 0$. The idea is to make a projection of the second column to L and apply the lemma to L. There will be no problems in this situation.

Consider an orthonormal system of k vectors, which are orthogonal to the first column and to L. Let us insert them into our matrices between the first and the second columns. We obtain the matrices specifying the same order, because these new columns annihilate all rational vectors, which need to be compared by the second column. Now, using admissible upper-triangle transformations, we can make these new columns orthogonal to all previous columns in the new matrix. Then we can apply the lemma. Hence, the second column is uniquely determined.

Continuing all these operations for the whole matrix, we obtain the algorithm, which gives us the canonical matrix. During the proof we obtain that the number of minimal columns is an invariant of the order. The numbers of columns, which we need to insert in the matrix after each step, are also invariants.

Let us prove the second part of the theorem. By the proof, if two matrices specify the same ordering, then their canonical matrix – *orthogonal normal form* – is uniquely determined. If two matrices specify different orderings, then their canonical matrices cannot be equal, since we do not change order during transformations.

Corollary 2. *There are interesting invariants of monomial orderings such that the minimal number of columns representing the order and the number of orthogonal vectors inserted in the matrix at each step in the process of proving the theorem.*

Corollary 3. *We can operate with canonical monomial matrices. Different canonical matrices provide different monomial orderings and we can distinguish monomial matrices.*

Let us illustrate the previous theorem. We shall show that temporary insertion of the columns in the monomial matrix is necessary. Consider the following

Example 3. Let us investigate two matrices:

$$A = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

They specify the same monomial ordering. Their first columns are equal. The monomials that cannot be distinguished by the first column differ only in the third component. Thus, the second columns of these matrices distinguish them.

Two columns of A and the second column of B form the basis of $\mathbb{R}^3$. That is why we cannot transform A using U-matrix. The dimension over $\mathbb{R}$ decreases by 1 and over $\mathbb{Q}$ by 2 after applying the first column. Hence, we can insert one orthogonal column in A and B. By the projection along this vector we obtain the second column of canonical matrix. (Let us first make A and B orthogonal.) The extended and canonical matrices are:

$$C' = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ \sqrt{2/3} & -\sqrt{2/3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1/\sqrt{3} & 0 \\ \sqrt{2/3} & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 4. We can canonically determine the additional columns. They are the solutions of a system of linear equations. Then, let the first vector be the extension of the free solution $1, 0, \ldots, 0$ (free variables), etc.

1.4 Consequences

Proposition 1. *The cardinality of all monomial orderings in a fixed number of variables is continuum.*

Proof. One can give an independent proof of this fact, but let us apply the results obtained above.

Note that each monomial ordering can be specified by a matrix in $M_n(\mathbb{R})$. Thus, the set of all monomial orderings has the cardinal type not greater than continuum. Let us prove the reverse inequality. Consider the family of monomial matrices:

$$M = \{C \in M_n(\mathbb{R}) \mid C = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}, \ a \in \mathbb{R} \setminus \mathbb{Q}\}.$$

This set is a continuum set and each M specify *different* ordering. This is the case due to Theorem 6, according to which we can represent each matrix in a canonical form, but the canonical monomial matrix is uniquely determined.

Proposition 2. *If a monomial matrix A which specifies a monomial ordering on monomials in n variables consists of k columns and $k < n$, then A has at least one irrational element.*

Proof. The proof uses the fact that k rational columns are always linearly dependent on $\mathbb{Q}$ ($k < n$). We can clear out denominators in the equation of a linear dependence. We can also represent each element in $\mathbb{Z}$ as the difference of two natural numbers. Thus, there exist two monomials which cannot be distinguished. This contradicts the definition of a monomial ordering.

1.5 The Applications

Corollary 4. *If we try to reduce the number of columns in a rational monomial matrix, we loose the ability to present the monomial ordering in the computer.*

Proposition 3. *There are monomial orderings which cannot be presented in the computer.*

Proof. Let us consider the example:

$$C = \begin{pmatrix} 1 & 0 \\ \pi & 0 \end{pmatrix}.$$

Using Corollary 1, we obtain the result: the first column of this matrix forms the minimal system of columns specifying this ordering. (First column distinguishes all monomials, but this is an invariant property of an ordering.) That is why this matrix cannot be represented in a two-column minimal form. But we noted above that rational matrices must contain at least $n = 2$ columns.

Remark 5. If the degrees of polynomials in our computations are restricted by some natural number, we can represent the irrational elements of a monomial matrix with the necessary precision in the computer and use this approximation in our particular task.

In [4], the fan and walk of Gröbner bases are discussed. These notions give us some information about the distribution of different reduced Gröbner bases. The set of monomial ideals generated by the leading terms of Gröbner bases of an ideal w.r.t. different monomial orderings is proved to be finite ([4, Lemma 2.6]).

So the set Gröbner bases of an ideal with fixed set of leading terms is proved to be finite.

To use a fixed Gröbner basis, we need to know the leading term of each member of the basis. "To use" means to do the reduction process and other applications of Gröbner bases. Let G be the Gröbner base w.r.t. a monomial ordering $<$. We have proved that $<$ can be represented by its canonical orthogonal matrix form. The first column in the matrix could appear to have irrational entries and cannot be represented precisely in the computer.

This problem can be solved by means of the Gröbner fan. The main idea has been formulated in [4], Theorem 2.7 (a) and (d), which says that the interior of the cone ${D_i}^*$ is a nonempty subset of $\mathbb{R}^n$. As defined in [4], ${D_i}^*$ consists of all vectors in $\mathbb{R}^n$ which can correctly distinguish the monomials of elements in the Gröbner basis of an ideal according to $<$. Union of all cones ${D_i}^*$ covers $\mathbb{R}^n_+$ (for complete definitions and assertions see [4]).

Hence, we can choose a rational vector from this cone and use it instead of the vector which has some irrational components. We shall call this vector irrational. First, we should compute the Gröbner basis of the ideal I and then use the approximation of the irrational vector.

On the other hand, to compute Gröbner bases, we can use rational vectors instead of irrational vectors. The idea is based on the Gröbner walk technique. First of all, we should choose a rational vector as near as possible to irrational one. Suppose that we have a precise representation of this irrational vector in the computer. Multiple application of this procedure for computing this vector can be very expensive. Hence, we cannot use it in the Gröbner bases computations, but we can construct a segment from the end of the rational vector and to the end of the irrational one.

Then, we should compute the Gröbner basis G using rational numbers and find the intersection points of our segment with the cones C_G. These points are the solutions to the linear equations in the parameter t on the segment

$$x(t)LT(g) = x(t)NL(g). \tag{12}$$

In these equations, $NL(g)$ denotes a nonleading term of g and t is a parameter on the segment. We should do this operation for each element g of the Gröbner basis G of I and for any its nonleading term.

We may approximately find the solutions to equations (12) under the condition that the approximate solution must be inside the segment. This work being completed, we choose a rational vector which is nearer to the irrational one than these solutions. In this way, either we obtain a rational vector which gives us the same Gröbner basis as the irrational one or there exist solutions to (12). In the latter case, we must recompute once the Gröbner basis.

Thus, we can use the monomial matrices with rational elements in many cases. Let us classify the admissible transformations for these orderings.

Theorem 7. *If the monomial matrices $A, B \in M_n(\mathbb{Q})$ specify the same monomial ordering $<$, then we can use only U-type transformations to walk from A to B. Only upper-triangle transformations with positive elements on the diagonal are admissible for these matrices.*

Proof. In the conditions of the theorem (it is important) we obtain that $C = A^{-1}B$ specify the lexicographic order. Let $a = \alpha A, b = \beta A$. We have: $\alpha < \beta \iff a <_{lex} b$. Then $a <_{lex} b \iff aA^{-1} < bA^{-1} \iff aA^{-1}B <_{lex} bA^{-1}B$.

Thus, according to Lemma 2, the matrix C is an upper-triangle matrix U with positive elements in the diagonal. Hence, $A^{-1}B = U$ and $B = AU$.

Remark 6. An easy example shows that previous theorem will became wrong if we omit the condition $A, B \in M_n(\mathbb{Q})$:

$$A = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}.$$

Thus, $A^{-1}B$ does not specify $<_{lex}$.

2 Differential Orderings

In the previous case we considered the monomials in a finite set of variables. But in differential algebra we often have to consider infinite sets of differential variables, which are well ordered by some rankings.

A *ranking* $<$ is a total ordering on a set of differential variables DV with the basic set of derivation operators $\delta_j,\ j \in \mathbb{N}_m$ and basic variables $y_i,\ i \in \mathbb{N}_0^n$, satisfying the following conditions:

1. the property of translation: $(\alpha, i) < (\beta, j) \iff (\alpha+\gamma, i) < (\beta+\gamma, j)$;
2. the property of positivity: $(\alpha, i) \geqslant (0, i)$.

The notation $(\alpha, i) = (\alpha_1, \ldots, \alpha_m, i)$ is used for the differential variable $\delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m} y_i$. Any ranking well orders the set DV. The proof of this fact and examples of rankings could be found in [3]. Some approaches to classifications of all rankings were made in [1] and [7]. Below we shall call differential variables simply variables.

Now let us construct the monomials in set DV and consider the orderings on them. We mean that only finitely many variables can occur in monomial. We begin with some definitions. If $y_i,\ i \in \mathbb{N}_0^n$, are basic variables and $\delta_j,\ j \in \mathbb{N}_m$, are basic derivation operators, then every monomial can be represented as $t = u_1^{d_1} \cdot \ldots \cdot u_s^{d_s}$, where $u_k = \theta_k y_{i_k} = \delta_1^{p_{k1}} \ldots \delta_m^{p_{km}} y_{i_k}$ are different determinates and d_k are their degrees. But we may consider DV as an infinite well-ordered set of algebraically independent variables. In other words, we may disregard the differentiations of variables. Denote by DM the set of all monomials.

The orderings on such sets of monomials applied in many practical problems should satisfy the following two **properties**:

1. the restriction of this ordering to variables must be a ranking, i.e. the properties of translation and positivity must hold;
2. this ordering must be consistent with multiplication by monomials: $t_1 \leqslant t_2 \iff t_1 s \leqslant t_2 s \ \ \forall\, s \in DM$.

Note that in [9] four additional properties of the so called *differential term-orderings* were formulated. But we shall prove the following result *without* using any additional conditions, which played an important role in [9].

Theorem 8. *The set DM is well ordered w.r.t. any linear order satisfying Properties 1 and 2.*

Proof. According to Property 1, the set DV is well ordered.

We prove the theorem by *reductio ad absurdum*. Suppose that there exists an infinite sequence $\{s_i\}$ strictly decreasing w.r.t. a given order $<$; i.e., $i < j \iff s_i > s_j$. We need to obtain a contradiction to this fact.

The following definitions and lemmas are needed below.

Definition 4. *Let s be a differential monomial, u be a differential variable. The tail of the monomial s w.r.t. u is the monomial $T_u(s)$ constructed of the variables from s which are strictly lower than u. In this definition we take these variables in corresponding powers. By analogy, the head $H_u(s)$ of s w.r.t. u is the monomial constructed of the variables which are not lower than u. And if $x < y$ are variables then the medium part of s is the monomial $M_{x,y}(s) = \frac{s}{H_x(s)T_y(s)} = \frac{H_y(s)}{H_x(s)} = \frac{T_x(s)}{T_y(s)}$.*

It is evident that $H_u(s)T_u(s) = s\ \forall s, u \in DM$.

Remark 7. This terminology does not contradict the common sense, since we write variables in monomials starting from the smallest. Then the head is the first part of a monomial and the tail is the last one.

Definition 5. *The hyperdegree $G(s)$ of the monomial s is the sum of its degree and the number of variables occurring in s. Note that $G(H_u(s)) + G(T_u(s)) = G(s)\ \forall s, u$. Also note that $2 \deg s \geqslant G(s) > \deg s\ \forall s \in DM$.*

Definition 6. *Let us imagine that we count variables in a monomial taking into account their powers. Let S and T be multisets of variables occurring in s and t respectively. We shall say that monomial s majorises monomial t iff there exists an injective map $\phi : T \to S$ such that $\forall t \in T\ t \leqslant \phi(t)$. It is clear that if s majorises t, then $s \geqslant t$. For example, if $x < y$, then xy majorises x^2.*

Remark 8. Note that if all variables in s are greater than any variable in t, and $\deg s \geqslant \deg t$, then s majorises t. It is evident that if t divides s then s majorises t.

Lemma 5. *Let $\{s_i\}$ be a sequence of differential monomials such that $\{G(s_i)\}$ is bounded. In other words, there exist $N, D \in \mathbb{N}$ such that every element $\{s_i\}$ depends on at most N variables, and the degree of each element does not exceed D. Then, this sequence cannot strictly decrease w.r.t. $<$.*

Proof. The proof is by induction on N.

Base of induction. Note that we can extract a subsequence of monomials from $\{s_i\}$ with the following property: the degree of every monomial is the same (say, equals to $K \leqslant D$). We can assume that elements $\{s_i\}$ already satisfy this property. The case $N = 1$ is clear: all elements of the sequence are variables raised to the power K: $u_1^K > u_2^K > \ldots > u_n^K > \ldots$. Hence, $u_1 > u_2 > \ldots > u_n > \ldots$, but this sequence cannot strictly decrease, because the set of variables DV is well ordered according to Property 1.

Step of induction. Suppose that the statement is proved for all $N' < N$ and is not valid for a sequence $\{s_i\}$, where s_i depends on N variables.

Let us use the inner induction on D. The case $D = 1$ corresponds to $N = 1$ and has been considered above. Let $D > 1$. Suppose there exist a variable u and a subsequence $\{s_{i_k}\}$ such that u occurs in each s_{i_k}. Then we can divide all monomials in this subsequence by u. We obtain a strictly decreasing sequence such that the degree of each monomial in this sequence is lower than $D - 1$. In this case we can use the inductive assumption. Hence, it is sufficient to consider the following case: for each variable u there exist only finitely many monomials in $\{s_i\}$ containing u.

We do not need the induction on D in this case. As above, we can suppose that all monomials in $\{s_i\}$ have the same degree K. Denote by R the set of the highest variables occurring in monomials $\{s_i\}$. Since the set $DV \supset R$ is well ordered, there exists a minimal element x in R. Assume that x corresponds to s_k. Cutting off the beginning of the sequence we may assume that $k = 1$. As above, we can suppose that no other element from $\{s_i\}$ contains x (we shall exclude the finite set of monomials which contain x). Let us consider two cases.

Case 1. There exists a monomial $s_q > s_1$ such that all variables occurring in s_q are greater than x, i.e. $H_x(s_q) = 1$. Obviously, in this case the variables of s_q are higher than the variables of s_1. Since the degrees of the monomials are the same, we obtain the contradiction $s_q \leqslant s_k$ which proves the theorem.

Case 2. Suppose that all monomials in the sequence except for s_1 can be written in the form $s_i = p_i q_i$, where the monomials $p_i = H_x(s_i)$ and $q_i = T_x(s_i)$ are non-trivial, depend on at most $N - 1$ variables and all variables in p_i are lower than x and variables in q_i are higher than x. We can apply the induction hypothesis to monomials q_i. Thus, in the set $\{q_i\}$ there exists a minimal element q_{m_1}. We have the following inequalities:

$$p_j q_j = s_j < s_{m_1} = p_{m_1} q_{m_1} \leqslant p_{m_1} q_j \quad \forall j > m_1,$$

hence, $p_j < p_{m_1}$ $\forall j > m_1$.

Consider the set $\{q_i \mid i > m_1\}$ and find in this set the minimal element q_{m_2}. We can write the similar inequalities for $m_2 > m_1$ and, in particular, for $p_{m_2} < p_{m_1}$

$$p_l q_l = s_l < s_{m_2} = p_{m_2} q_{m_2} \leqslant p_{m_2} q_l < p_{m_1} q_l \quad \forall l > m_2.$$

Thus, we have $p_l < p_{m_2} < p_{m_1}$ $\forall l > m_2$. Continuing this process, we obtain a strictly decreasing sequence $\{p_{m_k}\}$ of monomials depending on at most $N - 1$ variables and whose degrees are bounded. By the induction hypothesis, we have a contradiction, which completes the proof.

Lemma 6. *Let $z < x$ be variables from DV, $s, t \in DM$ be monomials and*

$$G(M_{z,x}(s)) > 2G(H_z(t)).$$

Then $H_x(s)$ majorises $H_z(t)$, and, in particular, $H_x(s) \geqslant H_z(t)$.

Proof. Since $H_x(s) = H_z(s) M_{z,x}(s)$, we get $M_{z,x}(s) \leqslant H_x(s)$ and $H_x(s)$ majorises $M_{z,x}(s)$. Thus, it is sufficient to prove that $M_{z,x}(s)$ majorises $H_z(t)$. We have

$$2\deg(M_{z,x}(s)) \geqslant G(M_{z,x}(s)) > 2G(H_z(t)) > 2\deg(H_z(t)),$$

and $\deg(M_{z,x}(s)) > \deg(H_z(t))$. To complete the proof, note that every variable occurring in $M_{z,x}(s)$ is greater than any variable occurring in $H_z(t)$ according to Definition 4.

Proof of Theorem 8. According to Lemma 5, it is sufficient to show that, for every strictly decreasing sequence $\{s_i\}$, the sequences $\{N_i\}$ (the number of variables in the ith monomial) and $\{D_i = \deg s_i\}$ are bounded. Moreover, it is sufficient to show that the sequence $\{G(s_i) = N_i + D_i\}$ is bounded.

Assume the contrary. Then we can extract a subsequence $\{s_{i_k}\}$ from $\{s_i\}$ such that $\{G(s_{i_k})\}$ is strictly increasing. Without loss of generality, we may suppose that $\{s_i\}$ already satisfies this property. We need the following proposition which we shall prove a bit later:

Proposition 4. *For all such sequences $\{s_i\}$ and for all $x \in DV$ we can extract a strictly increasing subsequence from the sequence $\{G(T_x(s_i))\}$.*

Remark 9. Notice that extracting a subsequence from $\{s_i\}$ does not change the validity of the properties mentioned above and this proposition. That means that we may work with any subsequence $\{s_{i_k}\}$ instead of $\{s_i\}$. We shall use this remark below.

We claim that this proposition is sufficient to prove the theorem. Indeed, denote by R the set of the highest variables occurring in $\{s_i\}$ as in Lemma 5. Let us also denote by y the minimal element in R. Let s be the corresponding monomial, and D be its degree. Cutting off the beginning of the sequence, we may assume that this monomial is the first one, $s = s_1$. According to the proposition, we can choose for $x = y$ the monomial s_m, such that $m > 1$ and $G(T_x(s_m)) \geqslant 2 \deg s$. We obtain that $\deg T_x(s_m) \geqslant \deg s$. Since all variables in $T_x(s_m)$ are higher than in s, we get that s_m majorises s. Thus, $s_m \geqslant s = s_1$ while $m > 1$, i.e. we obtain a contradiction.

Now let us prove the proposition.

Proof. We shall use the principle of transfinite induction. It can be formulated in the following way: *if (for all $x \in X$) the validity of a property $A(x)$ can be derived from the validity of the properties $A(y)$ for all $y < x$, then the property $A(x)$ is valid for all $x \in X$, where X is a well-ordered set.* We shall apply this induction to the set DV.

If x is the minimal element in X then the statement is valid. In fact, we assumed that $\{G(s_i)\}$ strictly increases, but $G(s_i) = G(T_x(s_i))$. Thus, the base of induction is completed.

Now suppose that the statement is valid for all $y < x$, but is not valid for x. More precisely, assume that for some $M \in \mathbb{N}$ and for all $m \in \mathbb{N}$ the inequality $G(T_x(s_m)) < M$ holds. Since $\{G(s_i)\}$ strictly increases and $G(s_i) = G(H_x(s_i)) + G(T_x(s_i))$, we can extract a subsequence $\{s_{i_k}\}$ from $\{s_i\}$ such that $\{G(H_x(s_{i_k}))\}$ strictly increases. We may assume that $\{s_i\}$ already satisfies this property.

Our aim is to transform that sequence in such a way that no element s_i contains x. Let us imagine that it is impossible to extract a subsequence with this property from $\{s_i\}$. Then we can extract a subsequence $\{s_{i_j}\}$ from $\{s_i\}$ such that $\{\deg_x s_{i_j}\}$ is constant or increases (maybe not strictly) and $\{s_{i_1}\}$ contains x. Dividing s_{i_j} by the highest power of x occurring in it, we obtain a sequence with the same properties as $\{s_i\}$. Indeed, this new sequence strictly decreases, because $\{s_i\}$ does. If the hyperdegrees of its elements are bounded, we immediately apply Lemma 5 and get a contradiction. Else we may again consider a subsequence with strictly increasing hyperdegrees. The hyperdegrees of the tails w.r.t. x have not been changed after dividing, and thus are bounded by assumption. As before, without loss of generality we may suppose that $\{s_i\}$ is exactly this sequence.

Now we have that x does not occur in the elements of $\{s_i\}$. Let $n_1 = 1$. Consider the highest variable z in $H_x(s_{n_1})$. Then $z < x$, and, hence, the proposition is valid for z. Thus, there exists an index $n_2 > n_1$ such that

$$G(T_z(s_{n_2})) > M + 2G(H_x(s_{n_1})).$$

At the same time, $G(T_x(s_{n_2})) < M$ by assumption. Hence,

$$G(M_{z,x}(s_{n_2})) = G(T_z(s_{n_2})) - G(T_x(s_{n_2})) > 2G(H_x(s_{n_1})).$$

Since z is the highest variable in $H_x(s_{n_1})$ and x does not occur in the sequence, we have $H_x(s_{n_1}) = H_z(s_{n_1})$. Applying Lemma 6 to the monomial s_{n_2}, we obtain that $H_x(s_{n_2})$ majorises $H_z(s_{n_1}) = H_x(s_{n_1})$. Let us denote $p_i = H_x(s_{n_i}),\ q_i = T_x(s_{n_i})$. Then $p_2 \geqslant p_1$, but $s_{n_2} = p_2 q_2 < p_1 q_1 = s_{n_1}$, because $n_2 > n_1$. This immediately implies that $q_2 < q_1$. Now we can repeat the same reasoning for the monomial s_{n_2} instead of the monomial s_{n_1}. We obtain that there exists a monomial $s_{n_3} = p_3 q_3,\ n_3 > n_2 > n_1$ such that $q_3 < q_2 < q_1$. Arguing as above, we construct a strictly decreasing infinite sequence $\{q_i\}$. But $q_i = T_x(s_{n_i})$, and $G(q_i) < M$ by assumption. We have a contradiction with Lemma 5 for the sequence $\{q_i\}$. Hence, the step of induction has also been completed.

It has been shown above that the proof of the theorem follows immediately.

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References

1. Carrà Ferro, G., Sit, W.Y.: On Term Orderings and Rankings. *Computational Algebra*. Marcel Dekker, New York, 31–77, 1994
2. Cox, D., Little, J., O'Shea, D.: *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer-Verlag, New York, 1998
3. Kolchin, E.R.: *Differential Algebra and Algebraic Groups.* Academic Press, New York, 1973
4. Mora, T., Robbiano, L.: Gröbner Fan of an Ideal, *J. Symb. Comp.* **6** (1988) 183-208
5. Robbiano, L.: Term orderings on the polynomial ring. In: *Proc. EUROCAL 85*, B.F. Caviness (Ed.), Springer Lect. Notes Comp Sci. **204** (1985) 513–517
6. Robbiano, L.: On the theory of graded structures, *J. Symb. Comp.* **2** (1986) 139–170
7. Rust, C., Reid, G.J.: Rankings of Partial Derivatives. In: *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, W.Küchlin (ed.), ACM Press, New York (1997) 9–16
8. Trevisan, G.: Classificazione dei semplici ordinamenti di un gruppo libero commutativo con N generatori. *Rend. Sem. Mat. Padova* **22** (1953) 143–156
9. Weispfenning, V.: Differential Term-Orders. In: *Proc. of the 1993 International Symposium on Symbolic and Algebraic Computation*, M. Bronstein (Ed.), ACM Press, New York (1993)