Pseudo-Inverse Matrices and Solutions Bounded on R of Linear and Nonlinear Systems

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Abstract. Conditions for existence of solutions bounded on R for a linear and nonlinear ordinary differential system are obtained under the assumption that the operator L defined by the corresponding unperturbed linearized homogeneous system is of Fredholm type. If L is a Fredholm operator with index zero or L is a Fredholm operator and, in addition, has an exponential trichotomy on R, we obtain the well-known results.

1 Introduction

Let us denote by $BC(J)$ the Banach space of continuous vector functions $x : J \to R^n$ bounded on an interval J with norm $\|x\| = \sup_{t\in J} |x(t)|$, and by $BC^1(J)$ the Banach space of vector functions $x : J \to R^n$ continuously differentiable on J and bounded together with their derivative and with norm $\|x\| = \sup_{t\in J} |x(t)|+ \sup_{t\in J} |\dot{x}(t)|$; J will usually denote one of intervals $R = (-\infty, +\infty), R_- = (-\infty, 0]$ or $R_+ = [0, +\infty)$. Consider the system

$$\dot{x} = A(t)x \tag{1}$$

with an $n \times n$ matrix $A(t)$ whose components are real functions, continuous and bounded on the whole line $R = (-\infty, +\infty)$: $A(\cdot) \in BC(R)$. It is known [1] that the system (1) has an exponential-dichotomy (e-dichotomy) on an interval J if there exists a projector $P(P^2 = P)$ and constants $K \geq 1$ and $\alpha > 0$ such that

$$\|X(t)PX^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \qquad t \geq s$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq Ke^{-\alpha(s-t)}, \quad s \geq t$$

for all $t, s \in J$; $X(t)$ is the normal $(X(0) = I)$ fundamental matrix of system (1).
Consider the problem about solutions $x : R \to R^n, x(\cdot) \in BC^1(R)$ bounded on R of the inhomogeneous system

$$\dot{x} = A(t)x + f(t), \quad f(\cdot) \in BC(R) \tag{2}$$

In the nonresonance case, where the homogeneous system (1) has an e-dichotomy on R, and so system (1) has only trivial solution bounded on R while the inhomogeneous system (2) has a unique solution bounded on R for each $f(\cdot) \in BC(R)$. The resonance case, where system (1) has nontrivial solutions bounded on R, was investigated by K. Palmer, which gave sufficient [2] and necessary [3] conditions for the Fredholm property of the considered problem. Let us define more exactly some results of well-known Palmer's lemma [2, p. 245], which will be used below for the investigation of weakly perturbed nonlinear systems.

2 Linear Systems

Let the system (1) have an e-dichotomy on R_+ and R_- with projectors P and Q, respectively. The general solution of (2) bounded on both half-lines R_+ and R_- is given by

$$x(t,\xi) = X(t)\begin{cases} P\xi + \int_0^t PX^{-1}(s)f(s)ds - \int_t^\infty (I-P)X^{-1}(s)f(s)ds, & t \geq 0; \\ (I-Q)\xi + \int_{-\infty}^t QX^{-1}(s)f(s)ds - \int_t^0 (I-Q)X^{-1}(s)f(s)ds, & t \leq 0. \end{cases} \tag{3}$$

Solution (3) will be bounded on R only if the vector constant $\xi \in R^n$ satisfies the condition

$$P\xi - \int_0^\infty (I-P)X^{-1}(s)f(s)ds = (I-Q)\xi + \int_{-\infty}^0 QX^{-1}(s)f(s)ds$$

so that the constant $\xi \in R^n$ is determined from the algebraic system

$$[P-(I-Q)]\xi = \int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_{0}^{\infty} (I-P)X^{-1}(s)f(s)ds\,. \qquad (4)$$

Let $D = P-(I-Q)$ be an $n\times n$ matrix, and let D^+ be an $n\times n$ matrix, which is a pseudo-inverse after

Moore–Penrose to D [5,6]. We will denote by $P_{N(D)}$ and $P_{N(D^*)}$ the $n\times n$ matrices - orthoprojectors ($P_{N(D)}^2 = P_{N(D)} = P_{N(D)}^*$; $P_{N(D^*)}^2 = P_{N(D^*)} = P_{N(D^*)}^*$) projecting R^n onto the kernel $\ker D = N(D)$ and cokernel $ker D^* = N(D^*)$ of the matrix D, where the symbol $*$ means the operation of transposition.

System (2) has the solutions bounded on R only if the algebraic system (4) is solvable over $\xi \in R^n$. For this it is necessary and sufficient that the right-hand side of system (4) belongs to the orthogonal complement $N^{\perp}(D^*) = R(D)$ to subspace $N(D^*)$. It follows that

$$P_{N(D^*)}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_{0}^{\infty} (I-P)X^{-1}(s)f(s)ds\right\} = 0. \qquad (5)$$

Thus, the general solution of system (2) bounded on R has the form (3) with constant $\xi \in R^n$, which is determined from (4) as follows:

$$\xi = D^+\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_{0}^{\infty} (I-P)X^{-1}(s)f(s)ds\right\} + P_{N(D)}c, \quad \forall c \in R^n. \qquad (6)$$

In other words only if the condition (5) is satisfied, the general solution of the system (2) bounded on the whole line R has the form

$$x(t,c) = X(t)\begin{cases} PP_{N(D)}c + \int_0^t PX^{-1}(s)f(s)ds - \int_t^{\infty}(I-P)X^{-1}(s)f(s)ds \\ +PD^+\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \quad t\geq 0; \\ (I-Q)P_{N(D)}c + \int_{-\infty}^t QX^{-1}(s)f(s)ds - \int_t^0(I-Q)X^{-1}(s)f(s)ds \\ +(I-Q)D^+\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \quad t\leq 0. \end{cases}$$

Since $DP_{N(D)} = 0$ [6, p. 90], then $PP_{N(D)} = (I-Q)P_{N(D)}$. Let $\dim N(L) = r$, then

$$r = \mathrm{rang}\,[PP_{N(D)}] = \mathrm{rang}\,[(I-Q)P_{N(D)}]$$

and vice versa. Let

$$[PP_{N(D)}]_r = [(I-Q)P_{N(D)}]_r$$

be an $n\times r$ matrix whose columns represent a complete set of r linearly independent columns of matrix $PP_{N(D)} = (I-Q)P_{N(D)}$. Then

$$X_r(t) = X(t)[PP_{N(D)}]_r = X(t)[(I-Q)P_{N(D)}]_r$$

is an $n\times r$ matrix whose columns represent a complete set of r linearly independent solutions of system (2) bounded on R .

Since $P_{N(D^*)}D = 0$ [6, p. 90], we have $P_{N(D^*)}Q = P_{N(D^*)}(I-P)$. Therefore, condition (5) is equivalent to one of the conditions

$$P_{N(D^*)}\int_{-\infty}^{\infty} QX^{-1}(s)f(s)ds = 0, \qquad (7)$$

$$P_{N(D^*)}\int_{-\infty}^{\infty} (I-P)X^{-1}(s)f(s)ds = 0.$$

Let

$$d = \text{rang}\,[P_{N(D^*)}(I-P)] = \text{rang}\,[P_{N(D^*)}Q] = \dim N(L^*).$$

Then each of conditions (7) consists only of d linearly independent conditions. Really, let $[Q^* P_{N(D^*)}]_d$ be an $n \times d$ matrix whose columns are d linearly independent columns of the matrix $[Q^* P_{N(D^*)}]$. Note that $X^{*-1}(t)$ is the fundamental matrix of the system

$$\dot{x} = -A^*(t)x, \tag{8}$$

adjoint to (1). System (8) is an e-dichotomies on R_+ with a projector $I - P^*$ and on R_- with a projector $I - Q^*$ [2, p. 246]. Then, as above,

$$H(t) = X^{*-1}(t)[Q^* P_{N(D^*)}] = X^{*-1}(t)[(I-P^*)P_{N(D^*)}]$$

is an $n \times n$ matrix whose columns are composed of n solutions bounded on R of the system (8); and hence

$$H_d(t) = X^{*-1}(t)[Q^* P_{N(D^*)}]_d = X^{*-1}(t)[(I-P^*)P_{N(D^*)}]_d$$

is an $n \times d$ matrix whose columns represent a complete set of d linearly independent solutions bounded on R of system (8) adjoint to (1). Hence $H_d^*(t)$ is an $d \times n$ matrix whose rows represent a complete set of d linearly independent solutions of system (8) bounded on R. Thus, the Palmer's lemma can be formulated as follows.

LEMMA. *Let system (1) have an e-dichotomy on R_+ and R_- with projectors P and Q, respectively. Then:*
a) an operator $L: \quad BC^1(R) \to BC(R)$ defined by

$$(Lx)(t) = \dot{x}(t) - A(t)x(t) \tag{9}$$

is a Fredholm operator and

$$\text{ind}\,L = \text{rang}\,[PP_{N(D)}] - \text{rang}\,[P_{N(D^*)}(I-P)] =$$
$$= \text{rang}\,[(I-Q)P_{N(D)}] - \text{rang}\,[P_{N(D^*)}Q] = r - d;$$

b)*the homogeneous system (1) has an $r-$ parametric set of solutions bounded on R*:

$$X_r(t)c_r = X(t)[PP_{N(D)}]_r c_r = X(t)[(I-Q)P_{N(D)}]_r c_r; \quad \forall c_r \in R^r;$$
$$(\ r = \text{rang}\,[PP_{N(D)}] = \text{rang}\,[(I-Q)P_{N(D)}]\ );$$

c) *system (8) adjoint to (1) has a $d-$parametric set of solutions bounded on R :*

$$H_d(t)c_d = X^{*-1}(t)[Q^* P_{N(D^*)}]_d c_d = X^{*-1}(t)[(I-P^*)P_{N(D^*)}]_d c_d, \quad \forall c_d \in R^d;$$
$$(\ d = \text{rang}\,[P_{N(D^*)}(I-P)] = \text{rang}\,[P_{N(D^*)}Q]\ );$$

d) $f \in \text{Im}(L)$ in the only case when:

$$\int_{-\infty}^{\infty} H_d^*(s)f(s)ds = 0; \tag{10}$$

e) *the inhomogeneous system (2) has an $r-$parametric set of solutions bounded on R and the general solution of the system (2) bounded on R can be written as*

$$x_0(t, c_r) = X_r(t)c_r + (G[f])(t), \quad \forall c_r \in R^r, \tag{11}$$

where

$$(G[f])(t) = X(t)\begin{cases} \int_0^t PX^{-1}(s)f(s)ds - \int_t^{\infty}(I-P)X^{-1}(s)f(s)ds \\ +PD^+\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \quad t \geq 0; \\ \int_{-\infty}^t QX^{-1}(s)f(s)ds - \int_t^0 (I-Q)X^{-1}(s)f(s)ds \\ +(I-Q)D^+\{\int_{-\infty}^0 QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds\}, \quad t \leq 0; \end{cases} \tag{12}$$

is the generalized Green operator for the problem of solutions of the system (2) bounded on the whole line R with properties:

$$(LG[f])(t) = f(t), t \in R; \quad (G[f])(0+0) - (G[f])(0-0) = \int_{-\infty}^{\infty} H^*(s)f(s)ds.$$

COROLLARY 1.

Suppose that the homogeneous system (1) has an e-dichotomy on R_+ and R_- with projectors P and Q, respectively, and such that $PQ = QP = Q$. In this case the system (1) has an exponential trichotomy [7, p. 363] on R and the inhomogeneous system (2) has at least one solution bounded on R for every $f \in BC(R)$ [7, p. 371], in other words this is the so-called weakly-regular case [8, p. 37], [9]. In this case the Lemma can be formulated as follows.

Let system (1) have an e-dichotomy on R_+ and R_- with projectors P and Q, respectively, and such that $PQ = QP = Q$. Then:
a) an operator $L: \; BC^1(R) \to BC(R)$ defined by (9) is a Fredholm operator and

$$\text{ind}\, L = \text{rang}\, [PP_{N(D)}] = \text{rang}[(I-Q)P_{N(D)}] = r;$$

b) *the homogeneous system (1) has an $r-$parametric set of solutions bounded on R*:

$$X_r(t)c_r = X(t)[PP_{N(D)}]_r c_r = X(t)[(I-Q)P_{N(D)}]_r c_r; \quad \forall c_r \in R^r;$$

$$(\; r = \text{rang}\, [PP_{N(D)}] = \text{rang}\, [(I-Q)P_{N(D)}] \;);$$

c) *system (8) adjoint to (1) has only the trivial solution bounded on R*;
d) $f \in Im(L)$ for all $f \in BC(R)$;
e) *the inhomogeneous system (2) has an $r-$ parametric set of solutions bounded on R and the general solution of the system (2) bounded on R can be written as*

$$x_0(t, c_r) = X_r(t)c_r + (G[f])(t), \quad \forall c_r \in R^r,$$

where : (G [f]) (t) is the generalized Green operator (12) for the problem of solutions of system (2) bounded on the whole line R with property:

$$(LG[f])(t) = f(t), t \in R; \quad (G[f])(0+0) - (G[f])(0-0) = 0.$$

Proof. Really, since

$$P_{N(D^*)}D = 0$$

and

$$DP = (P - (I-Q))P = QP = Q,$$

then

$$P_{N(D^*)}Q = P_{N(D^*)}DP = 0.$$

So, the necessary and sufficient condition (10) for the existence of solution of equation (2) bounded on R is satisfied for every $f \in BC(R)$.

COROLLARY 2.

Suppose that the homogeneous system (1) has an e-dichotomy on R_+ and R_- with projectors P and Q, respectively, and such that $PQ = QP = P$. In this case the system (8) has an exponential trichotomy on R and the inhomogeneous system (2) has only one solution bounded on R but for not every $f \in BC(R)$. In this case Lemma can be formulated as follows.

Let system (1) have an e-dichotomy on R_+ and R_- with projectors P and Q, respectively, and such that $PQ = QP = P$. Then:
a) an operator L is a Fredholm operator and

$$\text{ind}\, L = -\text{rang}\, [P_{N(D^*)}(I-P)] = -\text{rang}\, [P_{N(D^*)}Q] = -d;$$

b)*the homogeneous system (1) has only trivial solution bounded on* R:

$$(\ r = \text{rang}\,[PP_{N(D)}] = \text{rang}\,[(I-Q)P_{N(D)}] = 0 \ );$$

c) *system (8) adjoint to (1) has an* $d-$*parametric set of solutions bounded on* R :

$$H_d(t)c_d = X^{*-1}(t)[Q^*P_{N(D^*)}]_d c_d = X^{*-1}(t)[(I-P^*)P_{N(D^*)}]_d c_d, \quad \forall c_d \in R^d;$$

$$(\ d = \text{rang}\,[P_{N(D^*)}(I-P)] = \text{rang}[P_{N(D^*)}Q] \ );$$

d) $f \in Im(L)$ *only in the case where condition (10) holds for* $f \in BC(R)$;
e) the inhomogeneous system (2) has a unique solution bounded on R *and this solution can be written as* $x_0(t) = (G[f])(t)$, *where* $(G[f])(t)$ *is the generalized Green operator (12) for the problem of solutions of the system (2) bounded on the whole line* R *with properties:*

$$(LG[f])(t) = f(t), t \in R; \quad (G[f])(0+0) - (G[f])(0-0) = \int_{-\infty}^{\infty} H^*(s)f(s)ds.$$

Proof. Really, since $DP_{N(D)} = 0$ and $PD = P(P-(I-Q)) = PQ = P$, then

$$PP_{N(D)} = PDP_{N(D)} = 0.$$

So, $r = 0$ and the homogeneous system (1) has only the trivial solutions bounded on R, and the inhomogeneous system (2) has a unique solution bounded on R.

COROLLARY 3.

Suppose that the homogeneous system (1) has an e-dichotomy on R_+ and R_- with projectors P and Q, respectively, and such that $PQ = QP = P = Q$. In this case the system (1) has an exponential dichotomy on R and the inhomogeneous system (2) has only one solution bounded on R for every $f \in BC(R)$. In this case the Lemma can be formulated as follows.

Let system (1) have an e-dichotomy on R_+ *and* R_- *with projectors* P *and* Q, *respectively, and such that* $PQ = QP = P = Q$. *Then:*
a) an operator L *is a Fredholm and* $\text{ind}\, L = 0$;
b) *the homogeneous system (1) has only trivial solution bounded on* $R(\ r = 0 \ )$;
c) system (8) adjoint to (1) has only the trivial solution bounded on $R(\ d = 0)$;
d) $f \in \text{Im}\,(L)$ for all $f \in BC(R)$;
e) *the inhomogeneous system (2) has a unique solution bounded on* R, *and this solution can be written as* $x_0(t) = (G[f])(t)$, *where* $(G[f])(t)$ *is the Green operator (12)* $(P = Q, D^+ = D^{-1})$ *for the problem of solutions of system (2) bounded on the whole line* R *with properties:*

$$(LG[f])(t) = f(t), t \in R; \qquad (G[f])(0+0) - (G[f])(0-0) = 0.$$

These results are defined more exactly than the Palmer's lemma [2, p. 245] and give a new formula different from [4] for the calculation index of the operator L and will essentially be applied for obtaining the new existence conditions for the solutions bounded on the whole line of a weakly perturbed linear [10] and nonlinear [11] systems.

3 Nonlinear Systems

For a weakly nonlinear system

$$\dot{x} = A(t)x + f(t) + \varepsilon Z(x,t,\varepsilon) \tag{13}$$

let us find the conditions for the existence of solutions $x = x(t,\varepsilon)$ bounded on R

$$x(\cdot,\varepsilon) : R \to R^n, x(\cdot,\varepsilon) \in BC^1(R), x(t,\cdot) \in C[0,\varepsilon_0],$$

which turns, for $\varepsilon = 0$, into one of generating solutions $x_0(t, c_r)$ (11) of system (2). The nonlinear vector function $Z(x, t, \varepsilon)$ is such that:

$$Z(\cdot, t, \varepsilon) \in C^1[\ \|x - x_0\| \leq q\ ]; \quad Z(x, \cdot, \varepsilon) \in BC(R); \quad Z(x, t, \cdot) \in C[0, \varepsilon_0].$$

THEOREM 1 (NECESSARY CONDITION).
Assume that system (1) has an e-dichotomy on R_+ and R_- with projectors P and Q, respectively. Let system (13) have a solution $x(t, \varepsilon)$ bounded on R $x(\cdot, \varepsilon) : R \to R^n, x(\cdot, \varepsilon) \in BC^1(R), x(t, \cdot) \in C[0, \varepsilon_0]$, and $x(t, \varepsilon)$ turns, for $\varepsilon = 0$, into one of generating solutions $x_0(t, c_r)$(11) of system (2) with the vector constant $c_r = c_r^0 \in R^r$. Then the vector c_r^0 satisfies the equation

$$F(c_r^0) = \int_{-\infty}^{\infty} H_d^*(s) Z(x_0(s, c_r^0), s, 0) ds = 0. \tag{14}$$

Proof. Condition (10) for the existence of generating solutions $x_0(t, c_r)$ (11) bounded on R is assumed to be fulfilled. Considering the nonlinearity in (13) as inhomogeneity and applying the Lemma to (13), we obtain the following condition:

$$\int_{-\infty}^{\infty} H_d^*(s) Z(x(s, \varepsilon), s, \varepsilon) ds = 0.$$

Passing to the limit as $\varepsilon \to 0$ in the integral we come to the required condition (14).

By analogy with a case of the periodic problem [6, p. 184] it is natural to call equation (14) the equation for generating amplitudes of the problem about solutions of the system (13) bounded on the whole line R. If equation (14) has a solution, the vector constant $c_r^0 \in R^r$ determines that generating solution $x_0(t, c_r^0)$ to which the solution $x = x(t, \varepsilon)$ bounded on R

$$x(\cdot, \varepsilon) : R \to R^n, x(\cdot, \varepsilon) \in BC^1(R), x(t, \cdot) \in C[0, \varepsilon_0], x(t, 0) = x_0(t, c_r^0)$$

of the original problems (13) may correspond. If equation (14), however, has no solution, problem (13) has no solution bounded on R in the considered space. Since here and below all expressions are obtained in the real form, we speak about the real solutions of the equation (14), which may be algebraic or transcendental.

By changing the variables in (13) according to the relation

$$x(t, \varepsilon) = x_0(t, c_r^0) + y(t, \varepsilon),$$

we arrive at the problem of finding sufficient conditions for the existence of solution $y = y(t, \varepsilon)$ bounded on R

$$y(\cdot, \varepsilon) : R \to R^n, y(\cdot, \varepsilon) \in BC^1(R), y(t, \cdot) \in C[0, \varepsilon_0], y(t, 0) = 0$$

for the problem:

$$\dot{y} = A(t) y + \varepsilon Z(x_0(t, c_r^0) + y, t, \varepsilon). \tag{15}$$

Taking into account the continuous differentiability of a vector function $Z(x, t, \varepsilon)$ in x and its continuity in ε in the neighbourhood of a point $x_0(t, c_r^0), \varepsilon = 0$, we can select a term linear in y and terms of zero order in ε :

$$Z(x_0(t, c_r^0) + y, t, \varepsilon) = f_0(t, c_r^0) + A_1(t) y + R(y(t, \varepsilon), t, \varepsilon), \tag{16}$$

where

$$f_0(t, c_r^0) = Z(x_0(t, c_r^0), t, 0), \quad f_0(\cdot, c_r^0) \in BC(R);$$

$$A_1(t) = A_1(t, c_r^0) = \frac{\partial Z(x, t, 0)}{\partial x} |_{x = x_0(t, c_r^0)}, \quad A_1(\cdot) \in BC(R);$$

$$R(0, t, 0) = 0, \frac{\partial R(0, t, 0)}{\partial y} = 0, \quad R(y, \cdot, \varepsilon) \in BC(R).$$

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Regarding formally the nonlinearity $Z(x_0 + y, t, \varepsilon)$ in system (15) as an inhomogeneity and applying the Lemma to (15), we obtain the following representation of a solution bounded on R of system (15):

$$y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon).$$

In this expression, the unknown vector of constants $c = c(\varepsilon) \in R^r$ is determined from the condition type (14) of the existence of such solution for system (15) :

$$B_0 c = -\int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau, \tag{17}$$

where

$$B_0 = \int_{-\infty}^{\infty} H_d^*(\tau)A_1(\tau)X_r(\tau)d\tau$$

is a $d \times r$ matrix;

$$r = \mathrm{rang}\,[PP_{N(D)}] = \mathrm{rang}\,[(I-Q)P_{N(D)}], \quad d = \mathrm{rang}\,[P_{N(D^*)}(I-P)] = \mathrm{rang}\,[P_{N(D^*)}Q].$$

The unknown vector function $y^{(1)}(t,\varepsilon)$ is determined with the help of the generalized Green operator (12) from the relation:

$$y^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[Z(x_0(\tau,c_r^0) + y,\tau,\varepsilon)\right]\right)(t),$$

Let $P_{N(B_0)}$ be a $r \times r$ matrix - orthoprojector: $R^r \to N(B_0)$, and let $P_{N(B_0^*)}$ be an $d \times d)$ matrix - orthoprojector: $R^d \to N(B_0^*)$. Equation (15) is solvable with respect to $c \in R^r$ if and only if

$$P_{N(B_0^*)}\int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau = 0. \tag{18}$$

If

$$P_{N(B_0^*)} = 0,$$

then condition (18) always holds and equation (17) is solvable with respect to $c \in R^r$ up to an arbitrary vector constant $P_{N(B_0)}c \quad (\forall c \in R^r)$ from the null-space of matrix B_0 :

$$c = -B_0^+\int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau + P_{N(B_0)}c.$$

For finding one of the solutions $y = y(t,\varepsilon)$ bounded on R of problem (15)

$$y(\cdot,\varepsilon) : R \to R^n, y(\cdot,\varepsilon) \in BC^1(R), y(t,\cdot) \in C[0,\varepsilon_0], y(t,0) = 0$$

we arrive at the following operator system:

$$y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon), \tag{19}$$

$$c = -B_0^+\int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)]d\tau,$$

$$y^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[Z(x_0(\tau,c_r^0) + y,\tau,\varepsilon)\right]\right)(t).$$

The operator system (19) belongs to the class of systems [6, p. 188], for which solvability a simple iteration method is applicable, which converges for $\varepsilon \in [0,\varepsilon_*] \subseteq [0,\varepsilon_0]$. Really, system (19) can be rewritten as:

$$z = L^{(1)}z + Fz, \tag{20}$$

where $z = \mathrm{col}\,(y(t,\varepsilon), c(\varepsilon), y^{(1)}(t,\varepsilon))$ is a $(2n+r)$-dimensional column vector;

$L^{(1)}$ and F are linear and nonlinear operators bounded on R :

$$L^{(1)} = \begin{pmatrix} 0 & X_r & I_n \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad L_1 * = -B_0^+\int_{-\infty}^{\infty} H_d^*(\tau)A_1(\tau) * d\tau;$$

$$Fz = \mathrm{col}\left[0, \int_{-\infty}^{\infty} H_d^*(\tau)R(y(\tau,\varepsilon),\tau,\varepsilon)d\tau, \varepsilon G\left[Z(x_0(\tau,c_r^0)+y,\tau,\varepsilon)\right]\right].$$

By virtue of a structure of an operator $L^{(1)}$ with zero blocks on the principal diagonal and below it, there exists the operator $(I_s - L^{(1)})^{-1}$. System (20) may be transformed to the form

$$z = Sz, \qquad (S := (I_s - L^{(1)})^{-1}F, \quad s = 2n + r) \tag{21}$$

with the contraction operator S in a sufficiently small neighbourhood of point $x_0(t, c_r^0)$, $\varepsilon = 0$. For the solution of operator system (21) one of variants of a fixed point principle [12] is applicable for sufficiently small $\varepsilon \in [0, \varepsilon_*]$. Using a simple iteration method for finding a solution of the operator systems (17), and hence for finding solutions bounded on R of the original system (13), we arrive at the following result.

THEOREM 2 (SUFFICIENT CONDITION).

Assume that the weakly nonlinear system (13) satisfies the conditions stated above, and thus the corresponding generating linear system (2) has an r - parameter set (11) of generating solutions $x_0(t, c_r)$ bounded on R. Then, for every value of the vector $c_r = c_r^0 \in R^r$ that satisfies the equation for generating amplitudes (14), provided that the condition

$$P_{N(B_0^*)} \;=\; 0, \tag{22}$$

is satisfied, there exists at least one solution bounded on R of system (13). More exactly there exists an $\rho-$ parameter set of solutions bounded on R of the system (13)

$$x(t,\varepsilon) = \lim_{k\to\infty} x_k(t,\varepsilon) + X_r(t)P_{N(B_0)\rho}c_\rho, \quad \forall c_\rho \in R^\rho, c_\rho = c_\rho(\varepsilon), c_\rho(0) = 0.$$

These solutions $x(t,\varepsilon):\ x(t,\cdot) \in C[0,\varepsilon_0]$ turns, for $\varepsilon = 0$, into the generation solution $x(t,0) = x_0(t,c_r^0)$ (11) and $x_k(t,\varepsilon)$ can be determined by a simple iteration method convergent for $\varepsilon \in [0,\varepsilon_*] \subseteq [0,\varepsilon_0]$:

$$y_{k+1}^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[Z(x_0(\tau,c_r^0)+y_k,\tau,\varepsilon)\right]\right)(t) \tag{23}$$

$$c_k = -B_0^+ \int_{-\infty}^{\infty} H_d^*(\tau)[A_1(\tau)y_k^{(1)}(\tau,\varepsilon) + R(y_k(\tau,\varepsilon),\tau,\varepsilon)]d\tau,$$

$$y_{k+1}(t,\varepsilon) = X_r(t)c_k + y_{k+1}^{(1)}(t,\varepsilon),$$

$$x_k(t,\varepsilon) = x_0(t,c_r^0) + y_k(t,\varepsilon), \quad k = 0,1,2,...; \quad y_0(t,\varepsilon) = 0;$$

where $P_{N(B_0)\rho}$ is a $r \times \rho$ matrix whose columns represent a complete set of ρ linearly independent columns of an $r \times r$ matrix $P_{N(B_0)}$, $\rho = \mathrm{rang}\,P_{N(B_0)} = r - \mathrm{rang}\,B_0 = r - d$.

In the case when the number of $r = \mathrm{rang}\,[PP_{N(D)} = (I-Q)P_{N(D)}]$ linear independent bounded on R solutions of the system (1) is equal to the number of $d = rang[P_{N(D^*)}(I-P) = P_{N(D^*)}Q]$ linear independent bounded on R solutions of the system (8) adjoint to (1) from the condition $P_{N(B_0^*)} = 0$ we have $P_{N(B_0)} \;=\; 0$, and hence $det B_0 \neq 0$. In this case from the theorem 2 we shall receive the following statement [13].

THEOREM 3 (SUFFICIENT CONDITION).

Assume that the weakly nonlinear system (13) satisfies the conditions stated above, and thus the corresponding generating linear system (2) has an r - parameter set (11) of generating solutions $x_0(t, c_r)$ bounded on R. Then, for every value of the vector $c_r = c_r^0 \in R^r$ that satisfies the equation for generating amplitudes (14), provided that the condition

$$\det B_0 \neq 0, \qquad (r = d), \tag{24}$$

is satisfied, there exists a unique solution bounded on R of system (13). This solution $x(t,\varepsilon):\ x(t,\cdot) \in C[0,\varepsilon_0]$ turns, for $\varepsilon = 0$, into the generation solution $x(t,0) = x_0(t,c_r^0)$ (11) and can be determined by a simple iteration method (23) convergent for $\varepsilon \in [0,\varepsilon_*] \subseteq [0,\varepsilon_0]$.

4 Conclusion

Necessary estimates for ε_* and for the approximation error of iteration process can be obtained in the standard way [12].

Condition (24) means [6] that the constant $c_r^0 \in R^r$ is a simple root of equation (14) for generating amplitudes of the problem about solutions bounded on the whole line R of system (13). Using the techniques from [6, p. 193], with some simplifying assumptions, the method used in this paper can be extended to the case of multiple roots of equation (14).

If L is a Fredholm operator with index zero and in case $r = 1$, from Theorem 3 we obtain the well-known result of K. Palmer [2, p. 248]. If L is a Fredholm operator and, in addition, has an exponential trichotomy on R, from Theorem 2 we obtain the earlier known result of S. Elaidy and O. Hajek [7].

5 Examples

1. Let us consider the system

$$\dot{x} = A(t)x + f(t) + \varepsilon A_1(t)x, \tag{25}$$

where

$$A(t) = \operatorname{diag}\{-\tanh\ t, -\tanh\ t, \tanh\ t\}, \quad A_1(t) = \{a_{ij}(t)\}_{i,j=1}^3 \in BC(R).$$

We can easily verify that $X(t) = \operatorname{diag}\{2/(e^t + e^{-t}), 2/(e^t + e^{-t}), (e^t + e^{-t})/2\}$. The homogeneous system $\dot{x} = A(t)x$ is the e-dichotomies on both half-lines R_+ and R_- with projectors $P = \operatorname{diag}\{1, 1, 0\}$ and $Q = \operatorname{diag}\{0, 0, 1\}$. Then

$$D = 0, D^+ = 0, P_{N(D)} = P_{N(D^*)} = I_3; \quad r = \operatorname{rank} PP_{N(D)} = 2, d = \operatorname{rank} P_{N(D^*)}Q = 1;$$

$$X_r(t) = \begin{pmatrix} 2/(e^t + e^{-t}) & 0 \\ 0 & 2/(e^t + e^{-t}) \\ 0 & 0 \end{pmatrix}, \quad H_d(t) = \begin{pmatrix} 0 \\ 0 \\ 2/(e^t + e^{-t}) \end{pmatrix}.$$

The inhomogeneous system $\dot{x} = A(t)x + f(t)$ has a two-parametric set $x_0(t, c_r) = X_r(t)c_r + (G[f])(t)$, $\forall c_r \in R^2$ of solutions bounded on R only if the inhomogeneity $f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC(R)$ satisfies the following condition:

$$\int_{-\infty}^{+\infty} f_3(s)/(e^s + e^{-s})ds = 0, \qquad \forall f_1(t) \in BC(R), \quad \forall f_2(t) \in BC(R)$$

According to Theorems 1 and 2 we have the following result for system (25). For every value of the vector $c_r = c_r^0 \in R^2$ that satisfies the equation for generating amplitudes (14):

$$B_0 c_r^0 = -\int_{-\infty}^{\infty} H_d^*(s)A_1(s)(Gf)(s)ds,$$

provided that condition (22) $P_{N(B_0^*)} = 0$ is satisfied, there exists a one-parameter set of the solutions bounded on R of system (25) ($\rho = rangP_{N(B_0)} = r - rangB_0 = r - d = 1$). These solutions $x(t, \varepsilon)$: $x(t, \cdot) \in C[0, \varepsilon_0]$ turn, for $\varepsilon = 0$, into the generation solution $x(t, 0) = x_0(t, c_r^0)$, where

$$B_0 = \int_{-\infty}^{+\infty} H_d^*(t)A_1(t)X_r(t)dt = 4\int_{-\infty}^{+\infty} [a_{31}(t)/(e^t + e^{-t})^2, \ \ a_{32}(t)/(e^t + e^{-t})^2]dt.$$

If $a_{31}(t)$ and $a_{32}(t) \in BC(R)$ satisfy one of conditions

$$\int_{-\infty}^{+\infty} a_{31}(t)/(e^t + e^{-t})^2 dt \neq 0, \qquad \int_{-\infty}^{+\infty} a_{32}(t)/(e^t + e^{-t})^2 dt \neq 0,$$

then condition (22) is true. For example, if $a_{31}(t) = \text{Const} \neq 0$ or $a_{32}(t) = \text{Const} \neq 0$, then one of these inequalities is already realized, and condition (22) takes place. In this case the coefficients $a_{11}(t)$, $a_{12}(t)$, $a_{13}(t)$, $a_{21}(t)$, $a_{22}(t)$, $a_{23}(t)$, $a_{33}(t)$ can be arbitrary from the space $BC(R)$.

2. Let us consider the system (25), where

$$A(t) = \text{diag}\{-\tanh\ t, \tanh\ t\}, \quad A_1(t) = \{a_{ij}(t)\}_{i,j=1}^2 \in BC(R).$$

We can easily verify that $X(t) = \text{diag}\{2/(e^t+e^{-t}), (e^t+e^{-t})/2\}$, and the homogeneous system $\dot{x} = A(t)x$ is the e-dichotomies on both half-lines R_+ and R_- with projectors $P = \text{diag}\{1,0\}$ $Q = \text{diag}\{0,1\}$, respectively. Then

$$D = 0, D^+ = 0, P_{N(D)} = P_{N(D^*)} = I; \quad r = \text{rank}\, PP_{N(D)} = 1, d = \text{rank}\, P_{N(D^*)}Q = 1;$$

$$X_r(t) = \text{col}\{2/(e^t + e^{-t}), 0\}; \quad H_d^*(t) = \{0, 2/(e^t + e^{-t})\}.$$

The inhomogeneous system $\dot{x} = A(t)x + f(t)$ has a one-parametric set $x_0(t, c_r) = X_r(t)c_r + (G[f])(t)$, $\forall c_r \in R$ of solutions bounded on R only if $f(t) = \text{col}\{f_1(t), f_2(t)\} \in BC(R)$ satisfies the following condition:

$$\int_{-\infty}^{+\infty} f_2(s)/(e^s + e^{-s})ds = 0, \quad \forall f_1(t) \in BC(R).$$

According to Theorem 3 we have the following result for system (25) in this case. For every value of the constant $c_r = c_r^0 \in R$ that satisfies the equation for generating amplitudes (14):

$$B_0 c_r^0 = -\int_{-\infty}^{\infty} H_d^*(s)A_1(s)(Gf)(s)ds$$

provided that condition (24)

$$B_0 = \int_{-\infty}^{+\infty} H_d^*(t)A_1(t)X_r(t)dt = 4\int_{-\infty}^{+\infty} a_{21}(t)/(e^t + e^{-t})^2 dt \neq 0 \quad (r = d = 1),$$

is satisfied, there exists a unique solution bounded on R of the system (25). This solution $x(t,\varepsilon) : \ x(t,\cdot) \in C[0,\varepsilon_0]$ turns, for $\varepsilon = 0$, into the generation solution $x(t,0) = x_0(t, c_r^0)$.

For example, if $a_{21}(t) = \text{Const} = 1 \neq 0$ [8, p. 48], then the last inequality is already realized, and condition (24) takes place. In this case the coefficients $a_{11}(t)$, $a_{12}(t)$, and $a_{22}(t)$ can be arbitrary from the space $BC(R)$.

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