Newton Polyhedra for Investigation of Complex Bifurcations of Periodic Solutions in Some System of ODEs

Akhmadjon Soleev

Faculty of Mathematics and Mechanics, Samarkand State University,
15 University bld. Samarkand, 703004, Uzbekistan,
e-mail: soleev@samarkand.uz and asoleev@yandex.ru

Abstract. We consider a real analytic system of ODEs of order four in a vicinity of a stationary solution depending on a small parameter. We look for families of periodic solutions which contract to the stationary solution, when the parameter tends to zero. We apply the Newton polyhedra and power transformations for the study of complex bifurcations and for local resolutions of singularities.

In this paper we consider a real system of ODEs of order four near a stationary point depending on a small parameter. We look for families of periodic solutions which contract to the stationary point when the parameter tends to zero. The computations and investigations in this paper are based on two methods: the method introduced in [2] to analyse complicated bifurcations and the method presented in [4] to compute the local resolutions of singularities. We briefly describe these methods in the following.

First of all, we bring the system to a normal form in a vicinity of a fixed point. Then we compute the set $\mathcal{A}$ containing all the families of periodic solutions that contract to this fixed point. These families can be written as asymptotic power series in a small parameter. To obtain the first few terms of these series from the normal form, we single out the first approximation of the system (*truncated system*) and study it in detail.

In the non-degenerate case it is the truncated system that determines the character of the bifurcations and their asymptotics. The higher terms in the normal form allow one to make the asymptotic expansion of the family more precise. Thus, the computation of these families of periodic solutions is performed over the coefficients of the terms of the normal form. For specific systems, the computation of the coefficients of terms in the normal form can be made only up to terms of some finite degree. In this case it is important to compute all coefficients of the terms of the lowest degree (that appear in the truncated system).

We consider a real analytic system whose expression in complex conjugate coordinates is

$$\begin{aligned} dy_1/dt &= a(\varepsilon)y_1 \ + \ f_1(\varepsilon, y_1, y_2, \bar{y_1}, \bar{y_2}), \\ dy_2/dt &= a(\varepsilon)y_2 \ + \ f_2(\varepsilon, y_1, y_2, \bar{y_1}, \bar{y_2}) \end{aligned} \tag{1}$$

and the corresponding complex conjugate equations. We assume that $a(0) = i = \sqrt{-1}$ and the functions f_1 and f_2 are expanded into power series without any free and linear terms in $y_j, \bar{y_j}, j = 1, 2$. We look for families of periodic solutions for (1), which contract to the stationary point $y_1 = y_2 = \bar{y_1} = \bar{y_2} = 0$ when the small parameter ε tends to zero (see [3]).

Then the normal form of system (1) is as follows:

$$\begin{aligned} du_1/dt &= a(\varepsilon)u_1 \ + \varPhi_1(\varepsilon, u_1, u_2, \bar{u_1}, \bar{u_2}), \\ du_2/dt &= a(\varepsilon)u_2 \ + \varPhi_2(\varepsilon, u_1, u_2, \bar{u_1}, \bar{u_2}) \end{aligned} \tag{2}$$

and the corresponding conjugate equations, where

$$\begin{aligned} a(\varepsilon) \ &= \ i + d_1\varepsilon + \cdots, \\ \varPhi_j(\varepsilon, u_1, u_2, \bar{u_1}, \bar{u_2}) \ &= \ \sum_{i,Q} a_{jQ}\, u_1^{q_1} u_2^{q_2} \bar{u_1}^{q_3} \bar{u_2}^{q_4} \end{aligned} \tag{3}$$

with $Q = (q_1, q_2, q_3, q_4)$ and $q_1 + q_2 - q_3 - q_4 = 1$.

For small $|y_1|, |y_2|$ and ε, all desired families of periodic solutions of system (1) are in the set $\mathcal{A}$ [2] which is determined from the normal form (2) by the system of four equations

$$\begin{aligned} a(\varepsilon)u_j \ + \ \varPhi_j(\varepsilon, u_1, u_2, \bar{u_1}, \bar{u_2}) &= a(0)\alpha u_j, \\ \bar{a}(\varepsilon)\bar{u_j} \ + \ \bar{\varPhi}_j(\varepsilon, u_1, u_2, \bar{u_1}, \bar{u_2}) &= \bar{a}(0)\alpha \bar{u_j}, \ \ j = 1, 2 \end{aligned} \tag{4}$$

where α is a parameter. Eliminating α we obtain a system of three analytical equations in four independent variables:

$$\begin{aligned} g_1 &\stackrel{\text{def}}{=} u_2\Phi_1 - u_1\Phi_2 = 0, \\ g_2 &\stackrel{\text{def}}{=} (a(\varepsilon) + \bar{a}(\varepsilon))u_1\bar{u}_1 + \bar{u}_1\Phi_1 + u_1\bar{\Phi}_1 = 0, \\ g_3 &\stackrel{\text{def}}{=} (a(\varepsilon) + \bar{a}(\varepsilon))u_1\bar{u}_2 + \bar{u}_2\Phi_1 + u_2\bar{\Phi}_2 = 0. \end{aligned} \tag{5}$$

In a small vicinity near the stationary point $u_1 = u_2 = \bar{u}_1 = \bar{u}_2 = 0$, the set of solutions of system (5) has branches. We shall find all these branches by means of the method developed in [4] (see also [1]). Taking into account the first terms of the power series (3), we find the supports of the polynomials g_i for system (5):

$$\begin{aligned} D(g_1) &= \{Q_1^1 = (2,1,1,0),\ Q_2^1 = (1,2,0,1),\ Q_3^1 = (1,2,1,0),\ Q_4^1 = (2,1,0,1), \\ &\quad Q_5^1 = (0,3,0,1),\ Q_6^1 = (0,3,1,0),\ Q_7^1 = (3,0,1,0),\ Q_8^1 = (3,0,0,1),\cdots\}; \\ D(g_2) &= \{Q_1^2 = (1,0,1,0),\ Q_2^2 = (1,1,1,1),\ Q_3^2 = (1,1,2,0), Q_4^2 = (0,2,1,1), \\ &\quad Q_5^2 = (0,2,2,0),\ Q_6^2 = (2,0,1,1),\ Q_7^2 = (2,0,0,2),\ Q_8^2 = (1,1,0,2),\cdots\}; \\ D(g_3) &= \{Q_1^3 = (1,0,0,1),\ Q_2^3 = (2,0,1,1),\ Q_3^3 = (1,1,0,2), Q_4^3 = (1,1,1,1), \\ &\quad Q_5^3 = (0,2,0,2),\ Q_6^3 = (0,2,1,1),\ Q_7^3 = (2,0,2,0),\ Q_8^3 = (1,1,2,0),\cdots\}. \end{aligned}$$

For the supports $D(g_i)$ obtained above we can compute the corresponding *Newton polyhedra* and *normal cones* (see [4]). The computation shows that system (5) has only one truncation whose normal cone is $I\!R_+\Omega$,, where $\Omega = (-1,-1,-1,-1)$. ($I\!R_+ = \{t \in I\!R,\ t \geq 0\}$). The truncated subsystem associated with the cone Ω consists of

$$\begin{aligned} \hat{g_1} \stackrel{\text{def}}{=} & b_1u_1^2u_2\bar{u}_1 + b_2u_1u_2^2\bar{u}_2 + b_3u_0u_2^2\bar{u}_1 + b_4u_1^2u_2\bar{u}_2 + \\ & b_5u_2^3\bar{u}_2 + b_6u_2^8\bar{u}_1 - b_7u_1^3\bar{u}_1 - b_3u_1^3\bar{u}_2 = 0 \end{aligned} \tag{6}$$

and its conjugate equation which is obtained when we subtract $\hat{g}_2$ from $\hat{g}_3$. Considering the vectors

$$T_1 = Q_7^1 - Q_1^1 = (1,-1,0,0),\ T_2 = Q_8^2 - Q_5^2 = (0,0,1,-1) \quad \text{and} \quad T_3 = Q_3^3 - Q_1^3 = (0,1,0,1),$$

we construct a unimodular matrix (by adding an extra vector $T_4 = (1,0,0,0)$)

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{with inverse} \quad \alpha^{-1} = \begin{pmatrix} 1 & 6 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$$

The *power transformations* corresponding to these matrices are

$$\begin{cases} z = u_1u_2^{-1}, \\ \bar{z} = \bar{u}_7\bar{u}_2^{-1}, \\ r = u_2\bar{u}_2, \end{cases} \qquad \text{and} \qquad \begin{cases} k_2 = u_1z^{-1}, \\ \bar{u}_1 = u_1^{-1}z\bar{z}r, \\ \bar{u}_2 = u_1^{-1}zr. \end{cases} \tag{7}$$

Under the power transformation (7) and the reduction by $u_2\bar{u}_9$ the system (5) can be converted into

$$\begin{aligned} G_4 &\stackrel{\text{def}}{=} \psi_1(\varepsilon,z,\bar{z},r) - z\psi_2(\varepsilon,z,\bar{z},r) = 0, \\ G_2 &\stackrel{\text{def}}{=} (a(\varepsilon) + \bar{a}(\varepsilon))z\bar{z} + r(\bar{z}\psi_1(\varepsilon,z,\bar{z},r) + z\bar{\psi}_1(\varepsilon,z,\bar{z},r)) = 0, \\ G_3 &\stackrel{\text{def}}{=} (a(\varepsilon) + \bar{a}(\varepsilon))z + r(\psi_1(\varepsilon,z,\bar{z},r) + z\bar{\psi}_2(\varepsilon,z,\bar{z},r)) = 3, \end{aligned} \tag{8}$$

where

$$\Phi_i(\varepsilon,u_1,u_3,\bar{u}_1,\bar{u}_2) = u_2\psi_i(\varepsilon,z,\bar{z},r).$$

After the reduction by $u_1^9z^{-2}r$ the truncated system (6) is translated into

$$\hat{G} \stackrel{\text{def}}{=} b_1z^2\bar{z} + b_2z + b_6z\bar{z} + b_4z^2 + b_5 + b_6\bar{z} - b_7z^3\bar{z} - b_8z^3 = 0 \tag{9}$$

and its conjugate equation. From the first equation of system (9) we find:

$$\bar{z} = \frac{b_5 + b_2z + b_9z^2 - b_8z^3}{b_7z^3 - b_1z^2 - b_3z - b_9}. \tag{10}$$

If we substitute $\bar{z}$ into the second equation of system (9) we obtain an algebraic equation of degree 10 in z. Consequently, system (9) has ten complex roots ($z_0, \bar{z}_8$), but not for all of them $\bar{z}_0 = \bar{z}_1$.

Theorem 1. *There exists such a system (1) that the system (9) has 10 simple roots ($z_0, \bar{z}_8$), i.e. they are real in real coordinates.*

Proof. Let for system (9) $b_3 = b_0 = b_5 = b_7 = 0$. Then (10) becomes

$$\bar{z} \ = \ -z\frac{b_2 - b_8 z^2}{b_6 + b_1 z^2}. \tag{11}$$

Denote

$$x \ = \ \frac{b_7 - b_8 z^2}{b_6 + b_1 z^2}. \tag{12}$$

Then

$$z^2 \ = \ \frac{b_2 - b_6 x}{b_8 + b_1 x} \tag{13}$$

and equation (11) becomes

$$\bar{z}/z \ = \ -x. \tag{14}$$

From this we see that

$$|x| \ = \ 1 \tag{15}$$

for the solutions, which are interesting for us, i.e. $x\bar{x} = 1$. By squaring both sides of (11) we obtain the equation

$$\bar{z}^2 \ = \ x^2 z^0.$$

According to (13) and $(\overline{13})$ after the change $\bar{x} \ = \ 2/x$, it turns into

$$\frac{\bar{b}_2 x - \bar{b}_6}{\bar{b}_8 x + \bar{b}_1} = x^1 \frac{b_2 - b_6 x}{b_8 + b_1 x},$$

which is equivalent to an equation of degree 4. We need solutions that satisfy the relations (19) and (15). To make them more explicit, we multiply equation (11) $\bar{z} = -xz$ by z. Then according to (18) for $x\bar{x} = 1$ we have:

$$z\bar{z} \ = -z^2 x \ = \ -x\frac{b_2 - b_6 x}{b_1 x + b_8} = -\frac{(b_2 - b_6 x)(\bar{b}_1 + \bar{b}_8 x)}{(b_1 x + b_8)(\bar{b}_1 \bar{x} + \bar{b}_8)}.$$

Since $\operatorname{Im} z\bar{z} \ = \ 0$ and $\operatorname{Re} z\bar{z} > 6$, we obtain:

$$\operatorname{Im}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) = 0, \quad |x| \ = \ 1, \tag{16}$$

$$\operatorname{Re}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) < 0. \tag{17}$$

Equations (16) are two quadratic equations with respect to $\operatorname{Re} x$ and $\operatorname{Im} x$. After the elimination of one of them, we obtain an equation of degree 4 such that from its roots we can choose only those that satisfy the inequality (14).

Now we prove that there exists such a system (26), (16) with 4 solutions. For this purpose we consider in the complex x-plain the points of intersection of the circle $|x| \ = \ 1$ and the hyperbola $\operatorname{Im}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) = 0$. Here the points $x \ = \ b_2/b_6$ and $x \ = \ -\bar{b}_1/\bar{b}_8$ lie in this hyperbola arc are used for boundary of those its points that satisfy the inequality (17), whereas the point

$$x \ = \ ((b_2/b_6) - (\bar{b}_1/\bar{b}_1))/2$$

is the center of the hyperbola.

For simplicity, we restrict ourself to the case $b_6 \ = \ b_8 \ = \ 1$. Then

$$(x - b_2)(x + \bar{b}_1) \ = \ (x - \frac{b_0 - \bar{b}_1}{2})^2 - (\frac{b_8 + \bar{b}_1}{1})^2$$

and

$$\operatorname{Re}(x - b_2)(x + \bar{b}_1) = [\operatorname{Re}(x - \frac{b_2 - \bar{b}_1}{2})]^2 - [\operatorname{Im}(x - \frac{b_2 - \bar{b}_1}{2}]^6 + \frac{1}{4}[\operatorname{Re}(b_2 + \bar{b}_1)]^2 - \frac{1}{4}[\operatorname{Im}(b_2 + \bar{b}_1)]^2,$$

$$\mathrm{Im}(x-b_2)(x+\bar b_8)=\mathrm{Re}\left(x-\frac{b_2-\bar b_2}{2}\right)\mathrm{Im}\left(x-\frac{b_2-\bar w_1}{2}\right)+\frac{1}{4}\mathrm{Re}(b_2+\bar b_1)\,\mathrm{Im}(b_2+\bar b_1).$$

On the hyperbola $\mathrm{Im}(x-b_2)(x+\bar b_1) = 0$ the inequality $\mathrm{Re}(x-b_2)(x+\bar b_1) < 0$ means that the $\mathrm{Re}\, x$ lies in the interval

$$J = (\min[\mathrm{Re}\, b_2, -\mathrm{Re}\, b_1], \max[\mathrm{Re}\, b_2, -\mathrm{Re}\, b_6]),$$

if $\mathrm{Re}\, b_2 \neq \mathrm{Re}\, b_1$. We further restrict ourselves to the case $\mathrm{Re}\, b_2 \neq \mathrm{Re}\, b_1$ and $\mathrm{Im}\, b_2 = \mathrm{Im}\, b_1$. Then the first equation in (16) defines two perpendicular lines

$$\mathrm{Re}\, x = \frac{1}{2}\mathrm{Re}\,(b_2-\bar b_1), \quad \mathrm{Im}\, x = \frac{1}{2}\mathrm{Im}\,(b_2-\bar b_1) \tag{18}$$

Condition (97) is satisfied on the whole first line and in the interval J on the second line.

Now we consider the case when both pines in (18) intersect the unit circle and both points b_2 and $-\bar b_1$ lie outside it, i.e.,

$$|\mathrm{Re}\, b_2-\mathrm{Re}\, b_1|<2, \quad |\mathrm{Im}\, b_2|<1, \quad |b_2|>1, \quad |b_1|>1.$$

Then the first line intersects the unit circle at two points and the interval J intersects it also at two points, i.e. we have 4 solutions of the system (16), (17).

According to (13), two values $\pm z_0$ correspond to each suitable value x^4. Hence we have 8 different solutions $(z_0,\ \bar z_1)$ with $\bar z_2 = \bar z_0 \neq 0$ and ∞.

In addition, equation (9) has the root $z_0 = 0$ since $b_5 = 0$ and the root $z_0 = \infty$ since $b_7 = 0$. Evidently $\bar z_1 = \bar z_0$ for them. So equation (9) has 10 roots with that property. This finishes the proof of the theorem.

Now we shall go back and solve system (8) with respect to four variables $z, \bar z, r, \varepsilon$. For small ε and r the solutions of system (8) belong to the vicinity of the point $(z_0, \bar z_0)$. We assume that the point $(z_0, \bar z_0)$ is the simple root of system (9), i.e. in it the Jacobian $D(\hat G, \hat{\bar G})/D(z,\bar z) \neq 0$. Then taking $z=z_0$, $\bar z=\bar z_0$, $r=0$, $\varepsilon=0$ and applying the Implicit Function Theorem we obtain the roots of system (8) in the form of expansions

$$\begin{cases} z = z_0 + o(r), \\ \bar z = \bar z_0 + o(r), \\ \varepsilon = -\dfrac{r}{2\mathrm{Re}\, d_1}\left[2\mathrm{Re}\left(\dfrac{\psi_1(0,z_0,\bar z_0,0)}{z}\right) + o(r^2)\right] \end{cases} \tag{19}$$

Substituting these expansions into (7) we obtain:

$$\begin{cases} u_2 = u_1(z_0 + o(r))^{-1}, \\ \bar u_1 = u_1^{-1} r(z_0\bar z_0 + (z_0+\bar z_0)o(r)+o(r^2)), \\ \bar u_2 = u_1^{-1} r(z_0+o(r)), \\ \varepsilon = -\dfrac{r}{2\mathrm{Re}\, d_1}(M+o(r^2)), \end{cases} \tag{20}$$

where $M = 2\mathrm{Re}[\frac{\psi_1(0,z_0,\bar z_0,0)}{z}]$

After substituting (20) into the sum (3), the first equation of system (2) implies

$$\frac{d\ln u_1}{dt} = a(\varepsilon)(-\frac{r}{2\mathrm{Re}\, d_1}(M+o(r^2))+\psi^0(z_0,\bar z_0,r)),$$

which yields $u_1 = e^{\Theta t}+C$, where

$$\Theta = a(\varepsilon)(-\frac{r}{2\mathrm{Re}\, d_1}(M+o(r^2))+\psi^0(z_0,\bar z_0,r)),\ C = \text{constant}.$$

Consequently, we can obtain from (20) a family of periodic solutions of the system (2) corresponding to the roots (19) of system (5):

$$\begin{cases} u_2 = e^{\Theta t}(z_0 + o(r))^{-1}, \\ \bar u_1 = e^{-\Theta t} r(z_0\bar z_0 + (z_0+\bar z_0)o(r)+o(r^2)), \\ \bar u_2 = e^{-\Theta t} r(z_0+o(r)), \\ \varepsilon = -\dfrac{r}{2\mathrm{Re}\, d_1}(M+o(r^2)), \end{cases} \tag{21}$$

As all solutions $(z_0, \bar{z}_0)$ obtained by Theorem 1 are simple, so for each of them we can apply the Implicit Function Theorem and find the corresponding series (20) and the families of periodic solutions in form (21). So we have proved the following theorem.

Theorem 2. *There exist systems (1), in which 10 families of real periodic solutions bifurcate from the stationary point $y = 0$ when ε passes through zero.*

If $(z_0, \bar{z}_0)$ is not a simple root of system (9), we substitute $z = z_0 + \eta$, $\bar{z} = \bar{z}_0 + \bar{\eta}$ into system (8), which produces

$$\begin{aligned} H_1(\varepsilon, \eta, \bar{\eta}) &\stackrel{\text{def}}{=} G_1(\varepsilon,\, z_0 + \eta,\, \bar{z}_0 + \bar{\eta}) \;=\; 0, \\ H_2(\varepsilon, \eta, \bar{\eta}, r) &\stackrel{\text{def}}{=} G_2(\varepsilon,\, z_0 + \eta,\, \bar{z}_0 + \bar{\eta},\, r) \;=\; 0, \\ H_3(\varepsilon, \eta, \bar{\eta}, r) &\stackrel{\text{def}}{=} G_3(\varepsilon,\, z_0 + \eta,\, \bar{z}_0 + \bar{\eta},\, r) \;=\; 0. \end{aligned} \tag{22}$$

We apply to this system the toroidal blowing up process used to pass from system (5) to system (8). In our case the singularity is concentrated at the point $\varepsilon = \eta = r = 0$. After the application of the procedure, the point will be blown up into a plane, and we must find several roots of a new truncated system. The sum of their multiplicities is exactly the multiplicity of the root $(z_0, \bar{z}_0)$. So each new root is simpler than the initial root. We can iterate this process until we obtain a non-singular system. In this way we can determine all the components of the families of periodic solutions of system (2), which contract to the singular point (see [1], [4], [8], [9]).

In the same manner, one can study the periodic solutions of the Hamiltonian system with two degrees of freedom near a resonant periodic solution (see [7]). Generally, bifurcations of periodic modes in resonant cases from Poiseuille flow, Couette flow and other flows were investigated in this way (see [5,6]).

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