The Algebraic Problems Tied With Generalized Krylov – Bogolyubov Equation.

E.A. Grebenikov[1], M. Jakubiak[2], and D. Kozak–Skoworodkin[2]

[1] CC RAS, Moscow
[2] University of Podlasie, Siedlce, Poland

Abstract. The following scheme proved to be the most effective in the analysis and solvability of nonlinear equations. An optimal generating equation (the equation of first approximation) is constructed by an optimal smoothing operator. The initial iteration is defined by this generating equation. Then, the generalized Krylov – Bogolyubov equation is used to determine higher iterations. In this method the error of iterations does not depend on the error of the initial approximation, whereas in classic methods this is not true. This is associated with the fact that, in the former, a sequence of transformations of phase spaces is performed, and, for a given problem, an optimal phase space is found. By the methods of computer algebra, one can construct in the analytic form an asymptotic solution to a nonlinear resonant system of differential equations whose right-hand sides are multiple Fourier series.

1 The Generalized Krylov – Bogolyubov Equation

The modern constructive theory of nonlinear differential equations is based on various types of iterative methods, wich, as a rule, are not convergent, but are asymptotical in Poincaré sense. Divergence of iterative methods in nonlinear oscillation theory is created by frequent resonances, wich always appear in dynamical and technical models. The new results, concerning the Krylov – Bogolyubov method [1,2] take into account the new possibilities of computer algebra.

Suppose that we have differential equation

$$\frac{dz}{dt} = Z(z,t,\mu), \qquad z(0) = z_0, \tag{1}$$

where the function Z is determined in $G_{n+2} = \{z \in G_n, 0 \leq \mu << 1, 0 \leq t < +\infty\}$. Our problem is to construct a solution to (1).

The classic perturbations methods (Lagrange, Laplace, Poincaré, Lyapunov, ...) consist in the construction of iterative approximations that employ the ordinary differential equations to construct higher approximations.

Generally we have

$$\frac{dz}{dt} = \overline{Z}(z,t,\mu) + Z(z,t,\mu) - \overline{Z}(z,t,\mu),$$

$\overline{Z}$ is an arbitrary function, which is not yet defined.

Let us have

$$z(t,\mu) = \overline{z}(t,\mu) + u(t,\mu), \tag{2}$$

where $\overline{z}$ and "u" are some unknown functions. Problem (1) can be reduced to two Cauchy problems:

$$\frac{d\overline{z}}{dt} = \overline{Z}(\overline{z},t,\mu), \qquad \overline{z}(0) = \overline{z}_0, \tag{3}$$

and

$$\frac{du}{dt} = Z(\overline{z}+u,t,\mu) - \overline{Z}(\overline{z},t,\mu), \qquad u(0) = z_0 - \overline{z}_0, \tag{4}$$

Equation (3) defines the initial approximation $\overline{z}(t,\mu)$ of (1), and (4) defines the perturbation $u(t,\mu)$.

It is obvious from (4) that $u(t,\mu)$ depends not only on the initial function Z, but also on $\overline{z}$ and the new initial point $\overline{z}_0$.

Moreover, $u(t,\mu)$ can be determined only if the solution to (3) is known, i.e. after the determination of the initial approximation $\overline{z}(t,\mu)$ as a function of t. Thus, for (1) a set of variants of asymptotic theory with the parameters $\overline{Z}$ and $\overline{z}_0$ can be constructed.

Hence, $\overline{Z}$ and $\overline{z}_0$ can be interpreted as some generator of asymptotical theory for (1).

Common sense suggests that $\overline{Z}$ should be chosen in such a way that its analytic structure would be fairly simple, which is necessary to find the initial approximation $\overline{z}(t,\mu)$. At the same time, $\overline{Z}$ should retain the main properties of the function Z. The latter allows us to hope the solutions of (4) are "small". We need this to employ the iterative algorithms successfully.

Let us assume that the "perturbation" "u" depends on $\overline{z}, t, \mu$. Instead of (2), we write

$$z(t,\mu) = \overline{z}(t,\mu) + u(\overline{z}, t, \mu). \tag{5}$$

Equality (5) represents the transformation of the phase space $\{\overline{z}\}$ to $\{z\}$.

The following differential equation is obvious:

$$\frac{dz}{dt} = \frac{d\overline{z}}{dt} + \left(\frac{\partial u}{\partial \overline{z}}, \frac{d\overline{z}}{dt}\right) + \frac{\partial u}{\partial t}. \tag{6}$$

Symbol $\left(\frac{\partial u}{\partial \overline{z}}, \frac{d\overline{z}}{dt}\right)$ denotes the product of the Jacobi matrix $\frac{\partial u}{\partial \overline{z}}$ and the vector $\frac{d\overline{z}}{dt}$. Hence, instead of the equations for classical theory (3) and (4), we obtain equation (3) plus partial differential equation:

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial \overline{z}}, \overline{Z}(\overline{z}, t, \mu)\right) = Z(\overline{z} + u, t, \mu) - \overline{Z}(\overline{z}, t, \mu). \tag{7}$$

1 Equation (7) is called the generalized Krylov – Bogolyubov equation. The modern approach, which employs (3) and (7), differs from the classical one: the determination of $u(\overline{z}, t, \mu)$ from (7) does not require preliminary solution of equation (3). This allows us to determine the perturbation "u" and initial approximation $\overline{z}(t,\mu)$ independently.

Hence, the accuracies of their determination do not depend on each other. Moreover, the generating equation itself can be changed during the iterative process, i.e., it is possible that for the sth iteration, the sth generator is constructed (refined). This is impossible in the classical perturbation theory.

Bogolyubov was the first who strictly formalized the idea of mapping a class of nonlinear differential equations to the class of equations defined on a "smoothed" space. This equation first appeared in the work by Bogolyubov [3] when the question of the applicability of the averaging method to a special class of ordinary differential equations (the socalled standard differential equations) was considered. Condition (7) represents the formulation of the Cauchy problem for an n–dimensional system of partial differential equations of the first order with respect to the n–dimensional vector of perturbations u.

2 Asymptotic Solutions for Multifrequency Systems

Suppose that a mathematical model is described by a multifrequency system of the $(m+n)$th order

$$\frac{dx}{dt} = \mu X(x,y), x(0) = x_0, \quad \frac{dy}{dt} = \omega(x) + \mu Y(x,y), y(0) = y_0, \tag{8}$$

where x and X are m–dimensional vectors; y, Y and ω are n–dimensional vectors; and $\omega(x)$ is the frequency vector. Suppose that $X(x,y)$ and $Y(x,y)$ are 2π–periodic with respect to y. Under certain conditions [4], $X(x,y)$ and $Y(x,y)$ can be represented in the form of the n–fold Fourier series:

$$X(x,y) = \sum_{\|k\| \in I} X_k(x) e^{i(k,y)}, \quad Y(x,y) = \sum_{\|k\| \in I} Y_k(x) e^{i(k,y)},$$
$$i = \sqrt{-1}, \quad (k,y) = \sum_{s=1}^{n} k_s y_s, \quad \|k\| = \sum_{s=1}^{n} |k_s|, \quad I = (0,1,2,...), k_s = 0, \mp 1, ... \,. \tag{9}$$

We asymptotically expand (8) in powers of the small parameter μ. The generating system, which corresponds to (8), is chosen in the form

$$\frac{d\overline{x}}{dt} = \mu \overline{X}(\overline{x}, \overline{y}) + \sum_{k \geq 2} \mu^k A_k(\overline{x}, \overline{y}), \quad \frac{d\overline{y}}{dt} = \omega(x) + \mu \overline{Y}(\overline{x}, \overline{y}) + \sum_{k \geq 2} \mu^k B_k(\overline{x}, \overline{y}), \tag{10}$$

where $\overline{X}, \overline{Y}, A_k$, and B_k are, for the time being, the arbitrary functions of their arguments. Thus, (10) contains many arbitrary parameters and, therefore, can be considered as a multiparameter set of generators for the original nonlinear system.

We seek a change of variables (5) in the form of the formal series

$$x = \overline{x} + \sum_{k\geq 1} \mu^k u_k(\overline{x}, \overline{y}), \quad y = \overline{y} + \sum_{k\geq 1} \mu^k v_k(\overline{x}, \overline{y}), \tag{11}$$

where $u_k(\overline{x}, \overline{y})$ and $v_k(\overline{x}, \overline{y})$ are unknown functions.

If we differentiate (11) and use equations (8) and (10) to determine the functions of transformation u_k and v_k, then, instead of one quasilinear system (7), we obtain an infinite system of linear partial differential equations of the first order:

$$\begin{aligned}
&\left(\frac{\partial u_1}{\partial \overline{y}}, \omega(x)\right) = X(\overline{x}, \overline{y}) - \overline{X}(\overline{x}, \overline{y}), \\
&\left(\frac{\partial v_1}{\partial \overline{y}}, \omega(x)\right) = \left(\frac{\partial \omega}{\partial \overline{x}}, u_1\right) + Y(\overline{x}, \overline{y}) - \overline{Y}(\overline{x}, \overline{y}), \\
&\left(\frac{\partial u_k}{\partial \overline{y}}, \omega(x)\right) = F_k(\overline{x}, \overline{y}, u_1, v_1, ..., v_{k-1}, u_{k-1}, A_2, B_2, ..., A_k) \\
&\left(\frac{\partial v_k}{\partial \overline{y}}, \omega(x)\right) = \Phi(\overline{x}, \overline{y}, u_1, v_1, ..., V_{k-1}, u_k, A_2, B_2, ..., A_k, B_k) \quad k = 2, 3, ... \;.
\end{aligned} \tag{12}$$

For any vector index k, system (12) can be integrated in the analytic form [1] if functions $\overline{X}$ and $\overline{Y}$ are the mean values of X and Y functions.

Indeed, suppose that the generators $\overline{X}(\overline{x}, \overline{y})$ and $\overline{Y}(\overline{x}, \overline{y})$ are the partial sums of (9):

$$\overline{X}(\overline{x}, \overline{y}) = \sum_{\|k\| \in I_1} X_k(\overline{x}) e^{i(k, \overline{y})}, \quad \overline{Y}(\overline{x}, \overline{y}) = \sum_{\|k\| \in I_2} Y_k(\overline{x}) e^{i(k, \overline{y})}, \tag{13}$$

where I_1 and I_2 are the subsets of the set of all nonnegative integers I. In particular, I_1 or I_2 may consist of only a single number, zero. This implies that

$$\begin{aligned}
\overline{X}(x, y) = X_0(x) = \frac{1}{(2\pi)^n} \int_0^{2\pi} ... \int_0^{2\pi} X(x, y) dy_1 ... dy_n, \\
\overline{Y}(x, y) = Y_0(x) = \frac{1}{(2\pi)^n} \int_0^{2\pi} ... \int_0^{2\pi} Y(x, y) dy_1 ... dy_n.
\end{aligned} \tag{14}$$

Formulas (4) describe the averaging of the right-hand sides of (8) with respect to all fast variables $y_1, ..., y_n$ over the period; i.e., the free terms in the Fourier series (9) are taken as the mean values of $X(x, y)$ and $Y(x, y)$.

If X and Y are chosen according to (13), then

$$X(\overline{x}, \overline{y}) - \overline{X}(\overline{x}, \overline{y}) = \sum_{\|k\| \in I - I_1} X_k(\overline{x}) e^{i(k, \overline{y})}, \quad Y(\overline{x}, \overline{y}) - \overline{Y}(\overline{x}, \overline{y}) = \sum_{\|k\| \in I_2} Y_k(\overline{x}) e^{i(k, \overline{y})}.$$

By the method of characteristics and the classical theorem on the structure of a general solution to partial differential equations of the first order [5], one can find the exact solution of the first equations in (12). This solution is given by [1,2]

$$\begin{aligned}
u_1(\overline{x}, \overline{y}) &= \sum_{\|k\| \in I - I_1} \frac{X_k(\overline{x})}{i(k, \omega(\overline{x}))} e^{i(k, \overline{y})} + \varphi_1(\overline{x}), \\
v_1(\overline{x}, \overline{y}) &= \sum_{\|k\| \in I - I_2} \frac{Y_k(\overline{x})}{i(k, \omega(\overline{x}))} e^{i(k, \overline{y})} + \left[\frac{\partial \omega(\overline{x})}{\partial x}, \sum_{\|k\| \in I - I_1} \frac{X_k(\overline{x}) e^{i(k, \overline{y})}}{i^2(k, \omega(\overline{x}))^2}\right] \\
&+ \left(\left(\frac{\partial u_1}{\partial \overline{x}}, \varphi_1(\overline{x})\right), \overline{y}\right) + \psi_1(\overline{x}).
\end{aligned} \tag{15}$$

Here, φ_1 and ψ_1 are arbitrary differentiable functions of $\overline{x}_1, ..., \overline{x}_m$.

The integration of (12) for $k = 2, 3, ...$ is straightforward; hence, the functions $u_2, v_2, ...$ also admit analytical representation [12]. When determining the perturbations of the second order u_2 and v_2, we can freely dispose the functions $A_2, B_2, , \varphi_1$ and ψ_1.

It follows from (15) that $v_1(x, y)$ will increase approximately as a linear function of t for $\varphi_1 \neq 0$. Indeed, since $\overline{y}(t) \sim \omega(\overline{x})t$, the expression in (15)

$$\left(\left(\frac{\partial u_1}{\partial \overline{x}}, \varphi_1(\overline{x})\right), \overline{y}\right) \sim t \text{ for } \varphi_1(\overline{x}) \neq 0.$$

Since the first sums in the expression for $v_1(\overline{x}, \overline{y})$ are functions that are periodic with respect to $\overline{y}$, we obtain that $v_1(\overline{x}, \overline{y}) \sim t+$ periodic terms. Hence, in the second iteration, $u_2(\overline{x}, \overline{y}) \sim t^2$, $v_2(\overline{x}, \overline{y}) \sim t^3$, etc. Hence, we obtain the divergent iterative process. As numerical calculations show, we cannot eliminate all rapidly growing terms in the perturbations $u_k(\overline{x}, \overline{y})$ and $v_k(\overline{x}, \overline{y})$ by an appropriate choice of $A_k(\overline{x}, \overline{y})$ and $B_k(\overline{x}, \overline{y})$.

We draw the following important practical conclusion. The necessary condition for the perturbations $u_1, v_1, u_2, v_2, ...$ to be of oscillatory, rather than a rapidly growing, type is

$$\varphi_k(\overline{x}) \equiv 0, \psi_k(\overline{x}) \equiv 0, \ k = 1, 2, ... \, .$$

This means that the "best" perturbation theory is obtained when the generating equations and the equations for the perturbations are solved with the initial conditions $(\overline{x}_0, \overline{y}_0)$ that are different from the original ones (x_0, y_0). Indeed if $\varphi_1(\overline{x}_0) = \psi_1(\overline{x}_0) = 0$, then

$$u_1(\overline{x}_0, \overline{y}_0) = \sum_{\|k\| \in I - I_1} \frac{X_k(\overline{x}_0)}{i(k, \omega(\overline{x}_0))} e^{i(k, \overline{y}_0)} \neq 0,$$

$$v_1(\overline{x}_0, \overline{y}_0) = \sum_{\|k\| \in I - I_2} \frac{Y_k(\overline{x}_0)}{i(k, \omega(\overline{x}_0))} e^{i(k, \overline{y}_0)} + \left(\frac{\partial \omega(\overline{x}_0)}{\partial x_0}, \sum_{\|k\| \in I - I_1} \frac{X_k(\overline{x}_0) e^{i(k, \overline{y}_0)}}{i^2(k, \omega(\overline{x}_0))^2}\right) \neq 0,$$

and, with an accuracy of μ, the new initial conditions $(\overline{x}_0, \overline{y}_0)$ by the functional equations

$$x_0 = \overline{x}_0 + \mu u_1(\overline{x}_0, \overline{y}_0), \quad y_0 = \overline{y}_0 + \mu v_1(\overline{x}_0, \overline{y}_0).$$

Analogous equations with new initial conditions $(\overline{x}_0, \overline{y}_0)$ can be written for the perturbation theory of any order k:

$$x_0 = \overline{x}_0 + \sum_{s=1}^{k} \mu^s u_s(\overline{x}_0, \overline{y}_0), \quad y_0 = \overline{y}_0 + \sum_{s=1}^{k} \mu^s v_s(\overline{x}_0, \overline{y}_0). \tag{16}$$

This is the second important difference between the modern and classical perturbation theories. In the classical theory, it is difficult to dispose the choice of the initial point $(\overline{x}_0, \overline{y}_0)$. The new initial conditions $(\overline{x}_0, \overline{y}_0)$ can be determined from (16), e.g., by the method of simple iterations:

$$\overline{x}_0^{(j)} = x_0 - \sum_{s=1}^{k} \mu^s u_s(\overline{x}_0^{(j-1)}, \overline{y}_0^{(j-1)}), \quad \overline{y}_0^{(j)} = y_0 - \sum_{s=1}^{k} \mu^s v_s(\overline{x}_0^{(j-1)}, \overline{y}_0^{(j-1)}), \tag{17}$$

where $\overline{x}_0^{(0)} = x_0, \overline{y}_0^{(0)} = y_0$.

If we construct the perturbation theory of the second order, i.e., if we write (12) for $k = 2$, we obtain:

$$\left(\frac{\partial u_2}{\partial \overline{y}}, \omega(\overline{x})\right) = F_2(\overline{x}, \overline{y}, u_1, v_1, A_2), \ \left(\frac{\partial v_2}{\partial \overline{y}}, \omega(\overline{x})\right) = \varPhi_2(\overline{x}, \overline{y}, u_1, v_1, u_2, A_2, B_2).$$

In these equations, A_2 and B_2 are arbitrary functions. We choose them in such a way that

$$\int_0^{2\pi} ... \int_0^{2\pi} F_2 d\overline{y}_1 ... \overline{d} y_n = 0, \qquad \int_0^{2\pi} ... \int_0^{2\pi} \varPhi_2 d\overline{y}_1 ... d\overline{y}_n = 0. \tag{18}$$

These conditions ensure the choice of the solutions for u_2 and v_2 that also have an oscillatory, rather than a rapidly growing, character with respect to $\overline{y}$ and, consequently, with respect to t. This statement is true under the condition that φ_2 and ψ_2 (by analogy with φ_1 and ψ_1) are identically equal to zero. This procedure can be continued infinitely with respect to k.

The described algorithm consists in the successive change of variables

$$(x, y) \to (\overline{x}_1, \overline{y}_1) \to (\overline{x}_2, \overline{y}_2) \to ... \to (\overline{x}_s, \overline{y}_s),$$

where

$$x_s = \overline{x} + \sum_{k=1}^{s} \mu^k u_k(\overline{x}, \overline{y}), \quad y_s = \overline{y} + \sum_{k=1}^{s} \mu^k v_k(\overline{x}, \overline{y}).$$

The functions of transformation $u_k(\overline{x}, \overline{y})$ and $v_k(\overline{x}, \overline{y})$ are defined by exact analytical formulas of the form (15) with $\varphi_1(\overline{x}) = \psi_1(\overline{x}) \equiv 0$ and $A_k(\overline{x}, \overline{y})$ and $B_k(\overline{x}, \overline{y})$ defined by (8).

Naturally, the asymptotic solution to the Cauchy problem of the sth order requires the solution of the corresponding generating equation

$$\frac{d\overline{x}_s}{dt} = \mu\overline{X}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k A_k(\overline{x}_s), \quad \frac{d\overline{y}_s}{dt} = \omega(\overline{x}_s) + \mu\overline{Y}(\overline{x}_s, \overline{y}_s) + \sum_{k=2}^{s} \mu^k B_k(\overline{x}_s), \tag{19}$$

with the initial conditions $\overline{x}_s(0)$ and $\overline{y}_s(0)$. These conditions are determined by (17).

We use (11) to find the sth approximation to the exact solution of (8) in the form

$$\begin{aligned} x_s(t, \mu) &= \overline{x}_s(t, \mu) + \sum_{k=1}^{s} \mu^k u_k(\overline{x}_s(t, \mu), \overline{y}_s(t, \mu)), \\ y_s(t, \mu) &= \overline{y}_s(t, \mu) + \sum_{k=1}^{s} \mu^k v_k(\overline{x}_s(t, \mu), \overline{y}_s(t, \mu)). \end{aligned} \tag{20}$$

Note once again that in (20) u_k and v_k are determined by analytical methods. If the solution of the generating equation (20) can also be determined by analytical methods, then the asymptotic solution of the sth order (20) can be completely represented in the analytical form. It is these mathematical models that are widespread in cosmodynamics, for example, the three–dimensional Newton's three–body problem where the generator is a smoothing operator with respect to all variables y, the two–dimensional Newton's three–body problem where the generator is a smoothing operator under continuously acting perturbations, the motion of a television satellite around the earth, and so on. Analogous models are met in high–energy physics, geophysics, theory of waves, and other fields of science. If the generating equation cannot be integrated, we combine numerical methods for solving the generating equations with analytical ones in order to determine u_k and v_k. Thus, we obtain the most efficient (from the point of view of computation cost) methods for analyzing the original problem.

Finally, we will discuss the problems, which can be solved only by the advanced methods of computer algebra.

1) The construction of the averaging functions $\overline{X}(x, y), \overline{Y}(x, y)$.

First we calculate the initial frequencies $\omega_1(x_0), \omega_2(x_0), ..., \omega_n(x_0)$ and then we calculate the subsets of the integer numbers $I_1 \times I_2$, marking the proper k inequality vector

$$|(k, \omega(x_0))| < \varepsilon_1, \qquad |(k, \omega(x_0))| < \varepsilon_2.$$

If $\varepsilon_1 = \varepsilon_2$, $I_1 = I_2$. The ε_1 and ε_2 values are given a priori.

2) Afterwards, we calculate the perturbation of the first order $u_1(\overline{x}, \overline{y})$ and $v_1(\overline{x}, \overline{y})$ from equations (5).

3) The most arduous work is done while constructing F_2 and Φ_2 functions, thanks to which we can calculate the functions of the second approximation u_2 and v_2 from equations (12). It consists in multiplying Fourier series and assigning the resonant parts from the resulting products. Those resonant parts define the unknown functions A_2 and B_2.

4) If scientific researcher limits himself to the asymptotic theory of the second order, which is solving system (1) in form

$$\begin{aligned} x(t, \mu) &= \overline{x}(t, \mu) + \mu u_1(\overline{x}(t, \mu), \overline{y}(t, \mu)) + \mu^2 u_2(\overline{x}(t, \mu), \overline{y}(t, \mu)), \\ y(t, \mu) &= \overline{y}(t, \mu) + \mu v_1(\overline{x}(t, \mu), \overline{y}(t, \mu)) + \mu^2 v_2(\overline{x}(t, \mu), \overline{y}(t, \mu)), \end{aligned} \tag{21}$$

the initial conditions $\overline{x}(0,\mu)$and $\overline{y}(0,\mu)$ for solution of the generator system

$$\frac{d\overline{x}}{dt} = \mu\overline{X}(\overline{x},\overline{y}) + \mu^2 A_2(\overline{x}), \quad \frac{d\overline{y}}{dt} = \omega(\overline{x}) + \mu\overline{Y}(\overline{x},\overline{y}) + \mu^2 B_2(\overline{x}), \tag{22}$$

have to be calculated from nonlinear functional equations

$$\begin{aligned} \overline{x}(0,\mu) &= x(0,\mu) - \mu u_1(\overline{x}(0,\mu),\overline{y}(0,\mu)) - \mu^2 u_2(\overline{x}(0,\mu),\overline{y}(0,\mu)), \\ \overline{y}(0,\mu) &= y(0,\mu) - \mu v_1(\overline{x}(0,\mu),\overline{y}(0,\mu)) - \mu^2 v_2(\overline{x}(0,\mu),\overline{y}(0,\mu)). \end{aligned} \tag{23}$$

The solutions of the system of equations (23) are to be found only by means of iterative methods.

Finally, we will emphasize two extraordinary moments of the asymptotic theory based on averaging methods.

1. According to the super N.N. Bogolyubov's idea, a transformed equation (10) is not given a priori at the beginning, but is constructed at every step of calculations. This is meant to minimize the deviation of the asymptotic solution from the exact solution of the system (1). Such approach is not present in the classical perturbation theory.

2. The choice of the optimum initial conditions at every step of the constructing process improves the theory and the application practice of the resonant systems of differential equations.

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