Moduli Spaces of Low Dimension for Semi Brieskorn-Pham Singularities w.r.t. Right Equivalence

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Abstract. We provide equations for moduli spaces (of dimension ≤ 3) of semiquasihomogeneous singularities with fixed principal part, which we assume to be a Brieskorn-Pham polynomial. For dimension 4 and 5 we state only properties of the stabilizer and the invariant ring because the complexity of the calculation and/or the number of equations are too high. By these explicit computations we point out why a classification of more complicated singularities by means of unique normal forms fails due to the intrinsic complexity of moduli spaces.

Introduction

One of the important achievements of singularity theory was the classification of some "generic" classes of hypersurface singularities via normal forms by V.I. Arnold ([Arnold 1972]) like the classification of simple singularities in the families A_n, D_n, E_6, E_7, E_8, or more generally, singularities of functions with inner modality ≤ 4 (see [YS 1979] or Ch.15 in [AVG 1985]). For more complicated classes of singularities "the explicit determination of normal forms seems to be impossible and not appropriate" [1].

We provide empirical evidence of this claim by considering singularities of a class of functions, which is close to the classes of functions with inner modality ≤ 4. The functions of our interest are semiquasihomogeneous power series with principal part being a Brieskorn-Pham polynomial, which is special case of the more general definition of a class, where the principal part may be a quasihomogeneous polynomial with an isolated singularity at 0. For this class, in 1997, Greuel, Hertling, and Pfister came up with a geometric classification, i.e., they proved that for a given principal part f_0 a coarse moduli space $\mathcal{M}_{f_0}$ for semiquasihomogeneous singularities with principal part f_0 w.r.t. right equivalence exists (for contact equivalence one has to fix more invariants), cf. [GHP 1997]. Moduli spaces w.r.t. right equivalence (and fixed principal part) turn out to be quotients of affine varieties by finite groups, which are again affine varieties (cf., e.g., Ch. II.3.6 in [Kraft 1985]).

We calculate equations for all (but one) moduli spaces of dimension ≤ 3 for semiquasihomogeneous power series with fixed part being a Brieskorn-Pham polynomial by using the `qhmoduli.lib` (cf. [Bayer 2000]) of the computer algebra system Singular 2.0 (cf. [GPS01]). It will then be clear that unique normal forms are not appropriate for this and more complicated classes of singularities if the dimension of the moduli space exceeds 1.

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1 On Singularities and Their Classification

In this section we provide all necessary definitions and statements from singularity theory, we briefly mention the classification of singularities w.r.t. normal forms in the sense of Arnold, and we introduce the approach of Greuel, Hertling and Pfister for classifying singularities by means of moduli spaces because an approach by using normal forms seems to be inappropriate for more complicated classes.

[1] Quoted from the introduction of [GHP 1997].

1.1 On Singularities

In the sequel we denote by $\mathbf{C}\{x_1, x_2, \ldots, x_n\} =: \mathbf{C}\{\mathbf{x}\}$ the ring of convergent power series. Let $f : (\mathbf{C}^n, 0) \to \mathbf{C}$ be a holomorphic function defined on a small neighborhood of 0 (these functions correspond to convergent power series). Two functions $f, g \in \mathbf{C}\{\mathbf{x}\}$ are **right equivalent**, denoted by $f \sim_r g$, if there exists $\varphi \in Aut(\mathbf{C}\{\mathbf{x}\})$ s.t. $f = \varphi(g)$. A function $f \in \mathbf{C}\{\mathbf{x}\}$ has a **singularity** at 0 if $\frac{\partial f}{\partial x_1}(0) = \frac{\partial f}{\partial x_2}(0) = \ldots = \frac{\partial f}{\partial x_n}(0) = 0$. The singularity of f is called **isolated** iff there exists a neighborhood $U \subset \mathbf{C}^n$ of 0 s.t. 0 is the only singular point of f in U. Important algebraic tools for the study of the singularities of f are the **Jacobi ideal** of f, defined by $j(f) := \left\langle \frac{\partial f}{\partial x_i} : 1 \leq i \leq n \right\rangle$, the **Milnor algebra** of f, defined by $M_f := \mathbf{C}\{\mathbf{x}\}/j(f)$, and the **Milnor number** μ_f of f, which is the vectorspace dimension $\mu_f := \dim_{\mathbf{C}} M_f$ of the Milnor algebra. In the sequel we denote by B a monomial basis of M_f.

Remark 1. (a) Above we have provided a definition of the Milnor number in purely algebraic terms, but we like to mention that the Milnor number is a topological invariant. Originally, the Milnor number is defined by topological properties of a fibre of the Milnor fibration (cf. Ch. 2 in Vol. II of [AVG 1985]).
(b) The Milnor number gives an important characterization of isolated singularities, namely $\mu_f < \infty$ iff 0 is an isolated singularity of f.

In the sequel we fix a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbf{N}_+^n$. A monomial $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ is **quasihomogeneous of type** $(d; \mathbf{w})$ if $\deg_{\mathbf{w}}(\mathbf{x}^\alpha) := |\mathbf{x}^\alpha|_{\mathbf{w}} := \sum_{i=1}^n w_i \alpha_i = d$. A polynomial f is **quasihomogeneous of type** $(d; \mathbf{w})$ if all of its monomials are quasihomogeneous of type $(d; \mathbf{w})$. For a power series $0 \neq f \in \mathbf{C}\{\mathbf{x}\}$ we define $ord_{\mathbf{w}}(f) = \min\{|\mathbf{x}^\alpha|_{\mathbf{w}} : \mathbf{x}^\alpha$ appears in f with nonzero coefficient$\}$ and $ord_{\mathbf{w}}(0) := -1$. The power series f is called **semiquasihomogeneous of type** $(d; \mathbf{w})$ **with principal part** f_0 if f_0 is a quasihomogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at 0 and $f = f_0$ or $ord_{\mathbf{w}}(f - f_0) > d$. A polynomial $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ is called a **Brieskorn-Pham** polynomial if $f_0 = \sum_{i=1}^n x_i^{a_i}$ for $a_1, a_2, \ldots, a_n \in \mathbf{N}$. We call a power series which is right equivalent to a semiquasihomogeneous power series with principal part a Brieskorn-Pham polynomial a **semi Brieskorn-Pham singularity**.

For a morphism $\varphi \in Aut(\mathbf{C}\{x_1, x_2, \ldots, x_n)$ we define the **weighted order**, denoted by $ord_{\mathbf{w}}(\varphi)$, to be the maximal integer d s.t. $ord_{\mathbf{w}}(\varphi(x_i) - x_i) \geq w_i + d$ for $1 \leq i \leq n$.

1.2 On Normal Forms

A class K of singularities is a subset $K \subseteq \mathbf{C}\{\mathbf{x}\}$ which is invariant under the action of $Aut(\mathbf{C}\{\mathbf{x}\})$.

Arnold started the classification of singularities (cf. [Arnold 1972]) by providing normal forms for simple singularities[2] based on small deformations, and claimed that an "analogous classification of all critical points of codimension 6 is impossible, since here normal forms must inevitably contain parameters (moduli)". Simple singularities are given (up to stable equivalence) by the classes

A_k	D_k	E_6	E_7	E_8
x^{k+1}	$x^2y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

cf. [AVG 1985], vol. 1, p.184. For a quasihomogeneous polynomial f we define the inner modality $m(f) = |\{m \in B | \deg_{\mathbf{w}}(m) \geq \deg_{\mathbf{w}}(f)\}|$. Note that the normal forms of simple singularities have inner modality equal to 0 (and conversely, $m(f) = 0$ implies that f has a simple singularity). Up to now, quasihomogeneous singularities have been classified for inner modality $0, 1, 2, 3, 4$ (cf. Ch. 15 in [AVG 1985], [SAF 1986] and [YS 1979]).

1.3 On Moduli Spaces of Semiquasihomogeneous Singularities

For the more complicated class of semiquasihomogeneous singularities the normal forms contain parameters (with the exception of simple and simple elliptic singularities, cf. [Saito 1974]) which are, in general, not unique for a class. Therefore, [GHP 1997] started a classification by means of moduli spaces. Instead of providing the technically quite involved definition of a moduli space[3] since the existence of a moduli space is proved in [GHP 1997] (fine or coarse), we mention that the points of a moduli space are in

[2] A singularity is simple iff there exists a finite list of singularities s.t. each small deformation of f has a singularity out of this list.
[3] We refer to Ch. 1 of [HM 1998] for a description.

$1:1$ correspondence to isomorphism classes of the objects under consideration. Note that a moduli space depends on the equivalence relation employed for classification.

Given a quasihomogeneous polynomial f_0 of type $(d;\mathbf{w})$ with an isolated singularity at 0, the objects to classify are singularities defined by semiquasihomogeneous power series with principal part f_0 w.r.t. right equivalence, i. e., we consider the class $P_{f_0} = \{f \in \mathbf{C}\{\mathbf{x}\} \mid f \sim_r f_0 + f_1, ord_{\mathbf{w}}(f_1) > \deg_{\mathbf{w}}(f_0)\}$ which is invariant under the action of $Aut(\mathbf{C}\{\mathbf{x}\})$. For f_0 we define the set of **upper monomials** $B_- = \{m_1, m_2, \ldots, m_k\}$ where m_i is an element of a monomial basis of M_{f_0} s.t. $\deg_{\mathbf{w}}(m_i) > d$ and the space $T_- = \mathbf{C}^k$. In 1974, Arnold proved that the space T_- is a parameter space for the set P_{f_0} (i.e., there is a correspondence of points of T_- and isomorphism classes of P_{f_0}). For a constructive proof of this result we refer to Ch. 12.6, in [AVG 1985], p.209.

Theorem 1. *[Arnold 1974] Let f be a semiquasihomogeneous power series with principal part f_0. Then*

$$f \sim_r f_0 + \sum_{m_\alpha \in B_-} c_\alpha \cdot m_\alpha$$

for some $c_\alpha \in \mathbf{C}$, where B_- is the set of upper monomials of f_0.

Unfortunately, this correspondence is not unique which can be seen, e.g., by applying the $\mathbf{C}^*-$action, which is of order 0, of f_0 to F. Here the **order** of a morphism φ, denoted by $ord_{\mathbf{w}}(\varphi)$, is the maximal integer d s.t. $ord_{\mathbf{w}}(\varphi(x_i) - x_i) \geq w_i + d$ for $1 \leq i \leq n$ and $Aut_{\geq d}(\mathbf{C}\{\mathbf{x}\}) := \{\varphi \in Aut(R) : ord_{\mathbf{w}}(\varphi) \geq d\} \cup \{id\}$ for $d \geq 0$. For maps of order > 0 the correspondence is unique, as shown in Proposition 2.3 in [GHP 1997]. Additionally, [GHP 1997] proved that if f, g are semiquasihomogeneous power series of degree d and $\varphi(f) = g$ for $\varphi \in Aut(\mathbf{C}\{\mathbf{x}\})$ then $ord_{\mathbf{w}}(\varphi) \geq 0$ (Theorem 2.1 in [GHP 1997]). Therefore it suffices to consider the group $G^{\mathbf{w}}_{f_0} = \{\sigma \in Aut_{\geq 0}(\mathbf{C}\{\mathbf{x}\})/Aut_{\geq 1}(\mathbf{C}\{\mathbf{x}\}) \mid \sigma(f_0) = f_0\}$ in order to construct a moduli space.

Note that any morphism $\sigma \in G^{\mathbf{w}}_{f_0}$ induces a map $\Theta(\sigma) : T_- \to T_-$ by Arnold's Theorem. Now [GHP 1997] proved that Θ induces an algebraic group action on T_- when applied to F and that the group $E_{f_0} = \Theta(G^{\mathbf{w}}_{f_0})$, which acts algebraically on T_-, is finite. In general, the action of E_{f_0} is not linear, which complicates the computation of moduli spaces. The main result of [GHP 1997] for right equivalence is that, given f_0, the quotient

$$\pi : T_- \to T_-/E_{f_0} =: \mathcal{M}_{f_0}$$

is a coarse moduli space for semiquasihomogeneous singularities with principal part f_0 (w.r.t. right equivalence). The moduli space is fine if and only if $\mathcal{M}_{f_0} = \{0\}$. Since E_{f_0} is finite, the moduli space $\mathcal{M}_{f_0}$ is an affine variety and, in addition, a geometric quotient (i.e., the fibers of π correspond uniquely to the orbits of E_{f_0}). For the sake of completeness we state the theorem of Greuel, Hertling and Pfister (for definitions we refer to [GHP 1997]).

Theorem 2. *[GHP 1997]. Let $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0. The quotient*

$$\mathcal{M}_{f_0,\tau} := T_-/E_{f_0}$$

is a coarse moduli space for the functor $Unf^-_{f_0,\tau}$ where $\tau = alg/hol$.

Proof. We refer to Theorem 1.3 and Remark 1.5 in [GHP 1997].

An algorithm which, given f_0, computes equations for $\mathcal{M}_{f_0}$ is presented in [Bayer 2002] and an implementation is contained in the `qhmoduli.lib` of Singular 2.0 (cf. [Bayer 2000]).

2 On the construction of the moduli space

We briefly mention the algorithm for computing equations of moduli spaces of semiquasihomogeneous singularities introduced in [Bayer 2002]. Given the principal part $f_0 \in \mathbf{C}[t_1, t_2, \ldots, t_l]$, a coarse moduli space for semiquasihomogeneous singularities with principal part f_0 can be computed as follows.

1. Compute the set B_- of upper monomials of f_0.
2. Compute the stabilizer $G^{\mathbf{w}}_{f_0}$ of f_0.
3. Use Theorem 1.3.1 to compute the induced action α of $G^{\mathbf{w}}_{f_0}$ on T_-.
4. If necessary, linearize the action α.
5. Compute the induced group $E_{f_0} \subset Aut(T_-)$.

6. Compute a generating set of the invariant ring of E_{f_0} by means of Derksen's algorithm and the Reynolds operator.
7. Compute the embedding of T_-/E_{f_0} by using the generating set of step.

The algorithm is implemented in the computer algebra system SINGULAR 2.0 (cf. [GPS01]) and is contained in the `qhmoduli.lib` (cf. [Bayer 2000]). A description of Derksen's algorithm can be found in [Derksen 1997]. We illustrate the algorithm by computing a moduli space for $f_0 = x^3 + y^{12}$.

```
> LIB ''qhmoduli.lib'';
> ring R = 0,(x,y),ls; poly f = x3 + y12;
> ideal B = UpperMonomials(f); // step 1
> list stab = StabEqn(f); StabOrder(f); // step 2
36
> def R2 = ArnoldAction(f,stab,B); setring R2; // step 3
> def R3 = LinearizeAction(stabid, actionid,nvars(stab[1]));
> setring R3; // step 4; below: step 6 and 7
> def R4 = QuotientEquations(groupid,actionid,embedid); setring R4;
> imageid;
imageid[1]=Y(2)^2-Y(1)*Y(3) // defining equation in the ring C[Y(1),Y(2),Y(3)]
> dim(std(imageid));
2
// short version
> def T = ModEqn(f); setring T;
> dim(std(modid));
2
```

3 Semi Brieskorn-Pham Singularities

A polynomial $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ is called a **Brieskorn-Pham** polynomial if $f_0 = \sum_{i=1}^n x_i^{a_i}$ for $a_1, a_2, \ldots, a_n \in \mathbf{N}$. Brieskorn-Pham polynomials appeared, e.g., in the study of exotic spheres pursued by Milnor, Brieskorn and others in the 1960's. We consider the subclass of semiquasihomogeneous singularities with fixed principal part a Brieskorn-Pham polynomial because the computation of the moduli space may be much simpler due to the following two facts (proved in [Bayer 2002]):

- Generators for $G_{f_0}^{\mathbf{w}}$ (and E_{f_0}) can be written down without (by an easy) computation.
- The image of Θ is contained in $GL_k(\mathbf{C})$, i.e., the induced group action of E_{f_0} on T_- is linear.

Hence one may compute all elements of $G_{f_0}^{\mathbf{w}}$ or E_{f_0}, which are needed to compute the Reynolds operator, by matrix multiplication instead of computing the zeroset of a $0-$dim. ideal (the ideal of $G_{f_0}^{\mathbf{w}}$ or E_{f_0}). In general, the Reynolds operator must be used since Derksen's algorithm computes generators for the ideal generated by $\mathbf{C}[t_1, t_2, \ldots, t_m]_+^{G_{f_0}^{\mathbf{w}}}$, which are not necessarily invariant. This might be a significant improvement in the computation, see, e.g., Example 3.1 below.

Theorem 3. *Let $f_0 \in \mathbf{C}[x_1, x_2, \ldots, x_n]$ be a Brieskorn-Pham polynomial.*
(a) For $f_0 = \sum_{i=1}^n x_i^d$ the stabilizer is generated by all $n \times n$ permutation matrices and the diagonal matrix $\langle \exp(2\pi i/d), 1, \ldots, 1\rangle$, provided that $d > 2$.
(b) Let $f_0 = \sum_{i=1}^m f_0^{(d_i)}$ where $f_0^{(d_i)} = \sum_{j=r_{i-1}+1}^{r_i} x_j^{d_i}$ s.t. $d_i, r_i \in \mathbf{N}, 0 < d_i < d_j$ for $i < j$ and $r_0 = 0, r_i < r_{i+1}$. The stabilizer of f_0 is

$$G_{f_0}^{\mathbf{w}} = \bigoplus_{i=1}^m G_{f_0^{(d_i)}}^{\mathbf{w}}.$$

By using these properties of $G_{f_0}^{\mathbf{w}}$ it is shown in [Bayer 2002] that the induced group action of $G_{f_0}^{\mathbf{w}}$ (i.e., the action of E_{f_0}) on T_- is linear. For an example of an occurrence of a nonlinear action we refer to Example 4.1.4 in [Bayer 2002].

Theorem 4. *If f_0 is a Brieskorn-Pham polynomial then $E_{f_0} \subset GL_k(\mathbf{C})$, i.e., the induced action of $G_{f_0}^{\mathbf{w}}$ on T_- is linear.*

Still the computation of moduli spaces for semiquasihomogeneous power series with principal part a Brieskorn-Pham polynomial may be impossible due to the intrinsic complexity of the invariant ring of E_{f_0} and the quotient T_-/E_{f_0} which can be seen from the table in Section 4.4.

Example 1. Let $f_0 = x^3+y^3+z^7$ be a Brieskorn-Pham polynomial of type $(21;7,7,3)$. We have $\mu_{f_0}(0) = 24$, $|G_{f_0}^{(7,7,3)}| = |E_{f_0}| = 126$ and $T_- = \mathbf{C}^5$. The family

$$F(x,y,z,t_1,t_2,t_3,t_4,t_5) = f_0 + t_1xyz^5 + t_2yz^5 + t_3xz^5 + t_4xyz^4 + t_5xyz^3$$

contains all isomorphism classes of semi Brieskorn-Pham singularities with principal part f_0. The invariant ring $\mathbf{C}[t_1,t_2,t_3,t_4,t_5]^{E_{f_0}}$ is generated by 162 homogeneous invariants $h_1,h_2,\ldots,h_{162}$ with degrees ranging from 3 to 42. Among the 162 generators delivered by Derksen's algorithm, 107 are invariant. Even the computation of the zeroset of $G_{f_0}^{(7,7,3)}$ takes ca. 2h25' (without applying the Reynolds operator) but we can write down the generators of $G_{f_0}^{\mathbf{w}}$ and compute the induced action by hand (or Singular 2.0). Then all matrices of E_{f_0} are computed (12.68sec) and the Reynolds operator is applied to each non-invariant generator (1589.30sec). In order to compute the equations one has to eliminate the variables t_1,t_2,t_3,t_4,t_5 in the ideal $\langle y_1-h_1, y_2-h_2, ..., y_{162}-h_{162}\rangle$, which seems to be out of reach nowadays.

Remark 2. (a) Note that terms of the form $x_i^2+x_{i+1}^2+\ldots+x_r^2$ do not influence moduli spaces, because their partial derivative $2x_i, 2x_{i+1}\ldots, 2x_r$ do not lead to new upper monomials, so the group E_{f_0} remains unchanged. The group $G_{f_0}^{\mathbf{w}}$ indeed changes, its order might be equal to ∞ since any rotation leaves $x_i^2+x_{i+1}^2$ invariant.
(b) A Brieskorn-Pham polynomial $f_0 = \sum_{i=1}^n x_i^{a_i}$ is the direct sum of the functions $f^{(i)}(x) = x^{a_i}$ (cf. Section 2.7 (Vol. II) in [AVG 1985]). Hence several topological properties of Brieskorn-Pham singularities can be computed from the knowledge of these properties for the functions $f^{(i)}$. E.g., the Milnor number satisfies $\mu_{f_0} = \Pi_{i=1}^n \mu_{f^{(i)}}$ and the intersection matrix of f_0 can be computed by explicit formulae from $f^{(i)}$. Since the Milnor number for $F(\mathbf{x},t) = f_0 + \sum_{m_\alpha\in B_-} t_\alpha m_\alpha$ is constant if $t \in T_-$ varies (i.e., F is contained in the $\mu-$const stratum) several topological properties, like the intersection matrix, are the same for all $F(\mathbf{x},t), t\in T_-$.

4 List of Moduli Spaces

In this section we provide explicit descriptions for moduli spaces of semi Brieskorn-Pham singularities if their dimension is less than 4. For semi Brieskorn-Pham singularities whose moduli space is of dimension ≤ 5 we provide information about the groups $G_{f_0}^{\mathbf{w}}, E_{f_0}$, and about the embedding dimension[4] of their moduli spaces. Note that invariant rings and the embedding dimensions strongly depend on the representations of groups and not on the underlying groups. In general, different faithful representations of a group give rise to different invariant rings and different (minimal) embedding dimensions. Therefore we state generators for E_{f_0} which arise in practical computation.

In each table we denote by 'Type' the exponents of the Brieskorn-Pham polynomial f_0, by 'UM' all upper monomials of f_0, by $|G_{f_0}^{\mathbf{w}}|, |E_{f_0}|$ the order of the corresponding group, by 'Gen' generators for E_{f_0}, where $\xi_d = \exp(2\pi/d)$, by 'e-d' the embedding dimension of the moduli space and by 'nr' the number of equations of the moduli space. We describe $n\times n$ diagonal matrices by the elements $\langle d_1,d_2,\ldots,d_n\rangle$ of the diagonal.

All computations have been performed with the `qhmoduli.lib` ([Bayer 2000] or [Bayer 2002]) of Singular 2.0 (cf. [GPS01]). In cases where E_{f_0} contains permutations, invariant rings have been computed by algorithms of the `finvar.lib` ([Heydtmann 1997]). For the computation of the equations of these moduli spaces (by means of elimination) have employed the Hilbert-driven Gröbner bases algorithm (cf. [Traverso 1996]) of Singular 2.0.

Example 2. Consider the Brieskorn-Pham polynomial of type (4,4,4) (last row of dim = 4). We have $f_0 = x^4+y^4+z^4, F(x,y,z,t_1,t_2,t_3,t_4) = f_0 + t_1x^2y^2z^2 + t_2xy^2z^2 + t_3x^2yz^2 + t_4x^2y^2z$. The stabilizer $G_{f_0}^{\mathbf{w}}$ and the induced group E_{f_0} have order 384 and E_{f_0} is generated by the diagonal matrix $\langle -1,-1,-1,i\rangle$ and the permutation matrices of the cycles $(23),(234)$. The embedding dimension is 5 and the equation of $\mathcal{M}_{f_0}$ is given by $Y_1Y_4 - Y_5^2$.

[4] The embedding dimension of a quotient $\mathbf{C}^n/G \hookrightarrow \mathbf{C}^m$ is the dimension of the ambient space. It is given by the number of generators of the invariant ring of G, because they define the projection $\mathbf{C}^n \to \mathbf{C}^n/G$ and equations of the embedding can be computed by elimination. Since we choose minimal generating sets, we obtain the minimal embedding dimension.

4.1 Dimension = 1

We start the classification by considering semi Brieskorn-Pham singularities where the dimension of the moduli space is 1. Note that $G_{f_0}^{\mathbf{w}}$ is generated by permutation and diagonal matrices, hence the group E_{f_0} is cyclic. Since there is an isomorphism $\phi : T_-/E_{f_0} \to \mathbf{C}$ we can obtain a normal form for each class by $f_0 + t^d m, t \in \mathbf{C}, m \in B_-$ where d is the order of the (cyclic) group E_{f_0}.

Type	μ	UM	$\lvert G_{f_0}^{\mathbf{w}}\rvert$	$\lvert E_{f_0}\rvert$	Gen.
(3,7)	12	xy^5	21	21	$\langle\xi_{21}\rangle$
(3,8)	14	xy^6	24	12	$\langle\xi_{12}\rangle$
(3,9)	16	xy^7	27	9	$\langle\xi_9\rangle$
(4,5)	12	x^2y^3	20	10	$\langle\xi_{10}\rangle$
(4,6)	15	x^2y^4	24	6	$\langle\xi_6\rangle$
(5,5)	16	x^3y^3	50	5	$\langle\xi_5\rangle$
(3,3,4)	12	xyz^2	72	6	$\langle\xi_6\rangle$
(3,3,3,3)	16	$xyzt^2$	81	3	$\langle\xi_3\rangle$

4.2 Dimension = 2

Already in dimension 2, moduli spaces are too complicated for explicit normal forms. Note that the stabilizers are Abelian, but not necessarily cyclic.

Type	μ	UM	$\lvert G_{f_0}^{\mathbf{w}}\rvert$	$\lvert E_{f_0}\rvert$	E_{f_0}	e-d	nr
(3,10)	18	xy^8, xy^7	30	30	$\langle\xi_3,\xi_3\rangle, \langle\xi_{10}^7,\xi_5^4\rangle$	9	28
(3,11)	20	xy^9, xy^8	33	33	$\langle\xi_3,\xi_3\rangle, \langle\xi_{11}^7,\xi_{11}^8\rangle$	8	21
(3,12)	22	xy^{10}, xy^9	36	36	$\langle\xi_3,\xi_3\rangle, \langle\xi_{12}^9,\xi_{12}^{10}\rangle$	3	1
(4,7)	18	x^2y^5, x^2y^4	28	14	$\langle -1,-1\rangle, \langle\xi_7,\xi_7^5\rangle$	7	15
(4,8)	21	x^2y^6, x^2y^5	32	16	$\langle -1,-1\rangle, \langle\xi_8,-i\rangle$	3	1
(3,3,5)	16	xyz^3, xyz^2	90	15	$\langle\xi_3,\xi_3\rangle, \langle\xi_5,\xi_5^4\rangle$	7	15
(3,3,6)	20	xyz^4, xyz^3	108	18	$\langle\xi_3,\xi_3\rangle, \langle\xi_6,\xi_6^3\rangle$	2	0

Equations for the corresponding moduli spaces are given in the table below.

Type	Gen.
(3,10)	$Y_8^2 - Y_7Y_9, Y_7Y_8 - Y_6Y_9, Y_6Y_8 - Y_5Y_9, Y_5Y_8 - Y_3Y_9, Y_3Y_8 - Y_2Y_9, Y_2Y_8 - Y_1Y_9, Y_7^2 - Y_5Y_9, Y_6Y_7 - Y_3Y_9, Y_5Y_7 - Y_2Y_9, Y_3Y_7 - Y_1Y_9, Y_2Y_7 - Y_1Y_8, Y_6^2 - Y_2Y_9, Y_5Y_6 - Y_1Y_9, Y_3Y_6 - Y_1Y_8, Y_2Y_6 - Y_1Y_7, Y_5^2 - Y_1Y_8, Y_3Y_5 - Y_1Y_7, Y_2Y_5 - Y_1Y_6, Y_3^2 - Y_1Y_6, Y_2Y_3 - Y_1Y_5, Y_2^2 - Y_1Y_3, Y_1^2Y_8 - Y_4Y_9, Y_1^2Y_7 - Y_4Y_8, Y_1^2Y_6 - Y_4Y_7, Y_1^2Y_5 - Y_4Y_6, Y_1^2Y_3 - Y_4Y_5, Y_1^2Y_2 - Y_3Y_4, Y_1^3 - Y_2Y_4$
(3,11)	$Y_6^2 - Y_5Y_8, Y_5Y_6 - Y_4Y_8, Y_4Y_6 - Y_1Y_8, Y_5^2 - Y_1Y_8, Y_4Y_5 - Y_1Y_6, Y_4^2 - Y_1Y_5, Y_2^2 - Y_1Y_3, Y_1Y_2Y_6 - Y_3Y_8, Y_1^2Y_6 - Y_2Y_8, Y_1Y_2Y_5 - Y_3Y_6, Y_1^2Y_5 - Y_2Y_6, Y_1Y_2Y_4 - Y_3Y_5, Y_1^2Y_4 - Y_2Y_5, Y_3^3 - Y_2Y_7, Y_2Y_3^2 - Y_1Y_7, Y_1^2Y_2 - Y_3Y_4, Y_1^3 - Y_2Y_4, Y_1Y_3^2Y_6 - Y_7Y_8, Y_1Y_3^2Y_5 - Y_6Y_7, Y_1Y_3^2Y_4 - Y_5Y_7, Y_1^2Y_3^2 - Y_4Y_7$
(3,12)	$Y_2^2 - Y_1Y_3$
(4,7)	$Y_5^2 - Y_3Y_6, Y_3Y_5 - Y_2Y_6, Y_2Y_5 - Y_1Y_6, Y_4^2 - Y_1Y_7, Y_3^2 - Y_1Y_6, Y_2Y_3 - Y_1Y_5, Y_2^2 - Y_1Y_3, Y_1Y_4Y_5 - Y_6Y_7, Y_1^2Y_5 - Y_4Y_6, Y_1Y_3Y_4 - Y_5Y_7, Y_1Y_2Y_4 - Y_3Y_7, Y_1^2Y_4 - Y_2Y_7, Y_1^2Y_3 - Y_4Y_5, Y_1^2Y_2 - Y_3Y_4, Y_1^3 - Y_2Y_4$
(4,8)	$Y_2^2 - Y_1Y_3$
(3,3,5)	$Y_5^2 - Y_3Y_7, Y_3Y_5 - Y_1Y_7, Y_4^2 - Y_2Y_6, Y_2Y_4 - Y_1Y_6, Y_3^2 - Y_1Y_5, Y_2^2 - Y_1Y_4, Y_1Y_4Y_5 - Y_6Y_7, Y_1Y_2Y_5 - Y_4Y_7, Y_1^2Y_5 - Y_2Y_7, Y_1Y_3Y_4 - Y_5Y_6, Y_1^2Y_4 - Y_3Y_6, Y_1Y_2Y_3 - Y_4Y_5, Y_1^2Y_3 - Y_2Y_5, Y_1^2Y_2 - Y_3Y_4, Y_1^3 - Y_2Y_3$
(3,3,6)	0

4.3 Dimension = 3

There are 6 different Brieskorn-Pham polynomials s.t. the corresponding moduli space has dimension 3. We provide equations for all but one (722 equations). Since the order of $G_{f_0}^{\mathbf{w}}$ equals the order of E_{f_0}, we have omitted $\lvert G_{f_0}^{\mathbf{w}}\rvert$.

Type	μ	UM	$\lvert E_{f_0}\rvert$	E_{f_0}	e-d	nr
(3,13)	24	xy^{11}, xy^{10}, xy^9	39	$\langle \xi_{39}^7, \xi_{39}^4, \xi_{39}\rangle$	41	722
(3,14)	26	$xy^{12}, xy^{11}, xy^{10}$	42	$\langle \xi_{42}^8, \xi_{42}^5, \xi_{42}^2\rangle$	22	193
(3,15)	28	$xy^{13}, xy^{12}, xy^{11}$	45	$\langle \xi_3, \xi_3, \xi_3, \rangle, \langle \xi_{15}^7, \xi_{15}^9, \xi_{15}^2\rangle$	14	105
(5,6)	20	x^3y^4, x^2y^4, x^3y^3	30	$\langle \xi_{30}^8, \xi_{30}^2, \xi_{30}^3\rangle$	18	115
(6,6)	25	x^4y^4, x^3y^4, x^4y^3	72	$\langle \xi_6^4, \xi_6^3, \xi_6^4\rangle, \langle \xi_6^4, \xi_6^4, \xi_6^3\rangle, (23)$	4	1
(3,4,4)	18	xy^2z^2, xyz^2, xy^2z	96	$\langle \xi_6^4, \xi_6^3, \xi_6^4\rangle, \langle \xi_6^4, \xi_6^4, \xi_6^3\rangle, (23)$	11	55

Equations of moduli spaces are given in the table below.

Type	Equations
(3,13)	- 722 equations -
(3,14)	$Y_{21}^2 - Y_{20}Y_{22}, Y_{20}Y_{21} - Y_{15}Y_{22}, Y_{15}Y_{21} - Y_2Y_{22}, Y_{20}^2 - Y_2Y_{22}, Y_{15}Y_{20} - Y_2Y_{21},$ $Y_{17}^2 - Y_{14}Y_{18}, Y_{16}Y_{17} - Y_{13}Y_{18}, Y_{14}Y_{17} - Y_{11}Y_{18}, Y_{13}Y_{17} - Y_{10}Y_{18}, Y_{12}Y_{17} - Y_9Y_{18},$ $Y_{11}Y_{17} - Y_7Y_{18}, Y_{10}Y_{17} - Y_6Y_{18}, Y_9Y_{17} - Y_5Y_{18}, Y_8Y_{17} - Y_4Y_{18}, Y_3Y_{17} - Y_2Y_{18},$ $Y_{16}^2 - Y_{12}Y_{18}, Y_{14}Y_{16} - Y_{10}Y_{18}, Y_{13}Y_{16} - Y_9Y_{18}, Y_{12}Y_{16} - Y_8Y_{18}, Y_{11}Y_{16} - Y_6Y_{18},$ $Y_{10}Y_{16} - Y_5Y_{18}, Y_9Y_{16} - Y_4Y_{18}, Y_8Y_{16} - Y_3Y_{18}, Y_7Y_{16} - Y_6Y_{17}, Y_6Y_{16} - Y_5Y_{17},$ $Y_5Y_{16} - Y_4Y_{17}, Y_4Y_{16} - Y_2Y_{18}, Y_3Y_{16} - Y_1Y_{18}, Y_2Y_{16} - Y_1Y_{17}, Y_{15}^2 - Y_2Y_{20},$ $Y_{14}^2 - Y_7Y_{18}, Y_{13}Y_{14} - Y_6Y_{18}, Y_{12}Y_{14} - Y_5Y_{18}, Y_{11}Y_{14} - Y_7Y_{17}, Y_{10}Y_{14} - Y_6Y_{17},$ $Y_9Y_{14} - Y_5Y_{17}, Y_8Y_{14} - Y_4Y_{17}, Y_3Y_{14} - Y_2Y_{17}, Y_{13}^2 - Y_5Y_{18}, Y_{12}Y_{13} - Y_4Y_{18},$ $Y_{11}Y_{13} - Y_6Y_{17}, Y_{10}Y_{13} - Y_5Y_{17}, Y_9Y_{13} - Y_4Y_{17}, Y_8Y_{13} - Y_2Y_{18}, Y_7Y_{13} - Y_6Y_{14},$ $Y_6Y_{13} - Y_5Y_{14}, Y_5Y_{13} - Y_4Y_{14}, Y_4Y_{13} - Y_2Y_{17}, Y_3Y_{13} - Y_1Y_{17}, Y_2Y_{13} - Y_1Y_{14},$ $Y_{12}^2 - Y_3Y_{18}, Y_{11}Y_{12} - Y_5Y_{17}, Y_{10}Y_{12} - Y_4Y_{17}, Y_9Y_{12} - Y_2Y_{18}, Y_8Y_{12} - Y_1Y_{18},$ $Y_7Y_{12} - Y_5Y_{14}, Y_6Y_{12} - Y_4Y_{14}, Y_5Y_{12} - Y_2Y_{17}, Y_4Y_{12} - Y_1Y_{17}, Y_3Y_{12} - Y_1Y_{16},$ $Y_2Y_{12} - Y_1Y_{13}, Y_{11}^2 - Y_7Y_{14}, Y_{10}Y_{11} - Y_6Y_{14}, Y_9Y_{11} - Y_5Y_{14}, Y_8Y_{11} - Y_4Y_{14},$ $Y_3Y_{11} - Y_2Y_{14}, Y_{10}^2 - Y_5Y_{14}, Y_9Y_{10} - Y_4Y_{14}, Y_8Y_{10} - Y_2Y_{17}, Y_7Y_{10} - Y_6Y_{11},$ $Y_6Y_{10} - Y_5Y_{11}, Y_5Y_{10} - Y_4Y_{11}, Y_4Y_{10} - Y_2Y_{14}, Y_3Y_{10} - Y_1Y_{14}, Y_2Y_{10} - Y_1Y_{11},$ $Y_9^2 - Y_2Y_{17}, Y_8Y_9 - Y_1Y_{17}, Y_7Y_9 - Y_5Y_{11}, Y_6Y_9 - Y_4Y_{11}, Y_5Y_9 - Y_2Y_{14},$ $Y_4Y_9 - Y_1Y_{14}, Y_3Y_9 - Y_1Y_{13}, Y_2Y_9 - Y_1Y_{10}, Y_8^2 - Y_1Y_{16}, Y_7Y_8 - Y_4Y_{11},$ $Y_6Y_8 - Y_2Y_{14}, Y_5Y_8 - Y_1Y_{14}, Y_4Y_8 - Y_1Y_{13}, Y_3Y_8 - Y_1Y_{12}, Y_2Y_8 - Y_1Y_9,$ $Y_3Y_7 - Y_2Y_{11}, Y_6^2 - Y_5Y_7, Y_5Y_6 - Y_4Y_7, Y_4Y_6 - Y_2Y_{11}, Y_3Y_6 - Y_1Y_{11}, Y_2Y_6 - Y_1Y_7,$ $Y_5^2 - Y_2Y_{11}, Y_4Y_5 - Y_1Y_{11}, Y_3Y_5 - Y_1Y_{10}, Y_2Y_5 - Y_1Y_6, Y_4^2 - Y_1Y_{10}, Y_3Y_4 - Y_1Y_9,$ $Y_2Y_4 - Y_1Y_5, Y_3^2 - Y_1Y_8, Y_2Y_3 - Y_1Y_4, Y_7Y_{17}Y_{21} - Y_{16}Y_{22}, Y_7Y_{14}Y_{21} - Y_{13}Y_{22},$ $Y_6Y_{14}Y_{21} - Y_{12}Y_{22}, Y_7Y_{11}Y_{21} - Y_{10}Y_{22}, Y_6Y_{11}Y_{21} - Y_9Y_{22}, Y_5Y_{11}Y_{21} - Y_8Y_{22},$ $Y_7^2Y_{21} - Y_6Y_{22}, Y_6Y_7Y_{21} - Y_5Y_{22}, Y_5Y_7Y_{21} - Y_4Y_{22}, Y_4Y_7Y_{21} - Y_3Y_{22},$ $Y_2Y_7Y_{21} - Y_1Y_{22}, Y_7Y_{17}Y_{20} - Y_{16}Y_{21}, Y_7Y_{14}Y_{20} - Y_{13}Y_{21}, Y_6Y_{14}Y_{20} - Y_{12}Y_{21},$ $Y_7Y_{11}Y_{20} - Y_{10}Y_{21}, Y_6Y_{11}Y_{20} - Y_9Y_{21}, Y_5Y_{11}Y_{20} - Y_8Y_{21}, Y_7^2Y_{20} - Y_6Y_{21},$ $Y_6Y_7Y_{20} - Y_5Y_{21}, Y_5Y_7Y_{20} - Y_4Y_{21}, Y_4Y_7Y_{20} - Y_3Y_{21}, Y_2Y_7Y_{20} - Y_1Y_{21},$ $Y_7Y_{15}Y_{17} - Y_{16}Y_{20}, Y_4Y_7Y_{17} - Y_{15}Y_{18}, Y_2Y_7Y_{17} - Y_{15}Y_{16}, Y_7Y_{14}Y_{15} - Y_{13}Y_{20},$ $Y_6Y_{14}Y_{15} - Y_{12}Y_{20}, Y_7Y_{11}Y_{15} - Y_{10}Y_{20}, Y_6Y_{11}Y_{15} - Y_9Y_{20}, Y_5Y_{11}Y_{15} - Y_8Y_{20},$ $Y_7^2Y_{15} - Y_6Y_{20}, Y_6Y_7Y_{15} - Y_5Y_{20}, Y_5Y_7Y_{15} - Y_4Y_{20}, Y_4Y_7Y_{15} - Y_3Y_{20},$ $Y_2Y_7Y_{15} - Y_1Y_{20}, Y_4Y_7Y_{14} - Y_{15}Y_{17}, Y_2Y_7Y_{14} - Y_{13}Y_{15}, Y_1Y_7Y_{14} - Y_{12}Y_{15},$ $Y_4Y_7Y_{11} - Y_{14}Y_{15}, Y_2Y_7Y_{11} - Y_{10}Y_{15}, Y_1Y_7Y_{11} - Y_9Y_{15}, Y_1Y_6Y_{11} - Y_8Y_{15},$ $Y_4Y_7^2 - Y_{11}Y_{15}, Y_2Y_7^2 - Y_6Y_{15}, Y_1Y_7^2 - Y_5Y_{15}, Y_1Y_6Y_7 - Y_4Y_{15}, Y_1Y_5Y_7 - Y_3Y_{15},$ $Y_2^2Y_7 - Y_1Y_{15}, Y_5Y_7^2Y_{17} - Y_{18}Y_{20}, Y_5Y_7^2Y_{14} - Y_{17}Y_{20}, Y_5Y_7^2Y_{11} - Y_{14}Y_{20},$ $Y_5Y_7^3 - Y_{11}Y_{20}, Y_1^3Y_3Y_{20} - Y_7^2Y_{11}Y_{19}, Y_1^3Y_2Y_{20} - Y_7^3Y_{19}, Y_1^4Y_{20} - Y_6Y_7^2Y_{19},$ $Y_6Y_7^3Y_{17} - Y_{18}Y_{21}, Y_1^4Y_{16} - Y_{18}Y_{19}, Y_1^3Y_8Y_{15} - Y_7Y_{14}Y_{19}, Y_1^3Y_4Y_{15} - Y_7Y_{11}Y_{19},$ $Y_1^3Y_3Y_{15} - Y_6Y_{11}Y_{19}, Y_1^2Y_2^2Y_{15} - Y_7^2Y_{19}, Y_1^3Y_2Y_{15} - Y_6Y_7Y_{19}, Y_1^4Y_{15} - Y_5Y_7Y_{19},$ $Y_6Y_7^3Y_{14} - Y_{17}Y_{21}, Y_1^4Y_{13} - Y_{17}Y_{19}, Y_1^4Y_{12} - Y_{16}Y_{19}, Y_6Y_7^3Y_{11} - Y_{14}Y_{21},$ $Y_1^4Y_{10} - Y_{14}Y_{19}, Y_1^4Y_9 - Y_{13}Y_{19}, Y_1^4Y_8 - Y_{12}Y_{19}, Y_6Y_7^4 - Y_{11}Y_{21},$ $Y_1^4Y_6 - Y_{11}Y_{19}, Y_1^4Y_5 - Y_{10}Y_{19}, Y_1^4Y_4 - Y_9Y_{19}, Y_1^4Y_3 - Y_8Y_{19}, Y_1Y_2^4 - Y_7Y_{19},$ $Y_1^2Y_2^3 - Y_6Y_{19}, Y_1^3Y_2^2 - Y_5Y_{19}, Y_1^4Y_2 - Y_4Y_{19}, Y_1^5 - Y_3Y_{19}, Y_2^5Y_{21} - Y_{19}Y_{22},$ $Y_2^5Y_{20} - Y_{19}Y_{21}, Y_7^5Y_{17} - Y_{18}Y_{22}, Y_2^5Y_{15} - Y_{19}Y_{20}, Y_7^5Y_{14} - Y_{17}Y_{22},$ $Y_7^5Y_{11} - Y_{14}Y_{22}, Y_7^6 - Y_{11}Y_{22}, Y_2^6 - Y_{15}Y_{19}$
(3,15)	$Y_{12}^2 - Y_{11}Y_{13}, Y_8Y_{12} - Y_7Y_{13}, Y_6Y_{12} - Y_5Y_{13}, Y_5Y_{12} - Y_4Y_{13}, Y_4Y_{12} - Y_3Y_{13},$ $Y_2Y_{12} - Y_1Y_{13}, Y_8Y_{11} - Y_7Y_{12}, Y_6Y_{11} - Y_4Y_{13}, Y_5Y_{11} - Y_3Y_{13}, Y_4Y_{11} - Y_3Y_{12},$ $Y_2Y_{11} - Y_1Y_{12}, Y_9^2 - Y_3Y_{14}, Y_8^2 - Y_6Y_{10}, Y_7Y_8 - Y_5Y_{10}, Y_3Y_8 - Y_2Y_{10}, Y_7^2 - Y_4Y_{10},$ $Y_6Y_7 - Y_5Y_8, Y_5Y_7 - Y_4Y_8, Y_4Y_7 - Y_2Y_{10}, Y_3Y_7 - Y_1Y_{10}, Y_2Y_7 - Y_1Y_8,$ $Y_3Y_6 - Y_2Y_8, Y_5^2 - Y_4Y_6, Y_4Y_5 - Y_2Y_8, Y_3Y_5 - Y_1Y_8, Y_2Y_5 - Y_1Y_6, Y_4^2 - Y_1Y_8,$ $Y_3Y_4 - Y_1Y_7, Y_2Y_4 - Y_1Y_5, Y_2Y_3 - Y_1Y_4, Y_2Y_6Y_{13} - Y_{11}Y_{12}, Y_1Y_6Y_{13} - Y_{11}^2,$ $Y_1^2Y_{12} - Y_9Y_{13}, Y_1Y_9Y_{11} - Y_6^2Y_{14}, Y_1^2Y_{11} - Y_9Y_{12}, Y_6^2Y_9 - Y_1Y_3Y_{11},$ $Y_1Y_4Y_9 - Y_8Y_{14}, Y_3^2Y_9 - Y_{10}Y_{14}, Y_1Y_3Y_9 - Y_7Y_{14}, Y_2^2Y_9 - Y_6Y_{14},$ $Y_1Y_2Y_9 - Y_5Y_{14}, Y_1^2Y_9 - Y_4Y_{14}, Y_6^2Y_8 - Y_{10}Y_{13}, Y_5Y_6Y_8 - Y_{10}Y_{12},$ $Y_4Y_6Y_8 - Y_{10}Y_{11}, Y_2Y_6Y_8 - Y_7Y_{11}, Y_1^2Y_7 - Y_8Y_9, Y_6^3 - Y_8Y_{13}, Y_5Y_6^2 - Y_7Y_{13},$ $Y_4Y_6^2 - Y_7Y_{12}, Y_2Y_6^2 - Y_3Y_{13}, Y_1Y_6^2 - Y_3Y_{12}, Y_1Y_5Y_6 - Y_3Y_{11}, Y_2^2Y_6 - Y_1Y_{11},$ $Y_1^2Y_5 - Y_6Y_9, Y_1^2Y_4 - Y_5Y_9, Y_3^3 - Y_9Y_{10}, Y_1Y_3^2 - Y_7Y_9, Y_1^2Y_3 - Y_4Y_9, Y_1^3 - Y_2Y_9,$ $Y_3Y_{11}^3 - Y_8Y_9Y_{13}^2, Y_1^2Y_2Y_6 - Y_9Y_{11}, Y_7Y_{11}^4 - Y_9Y_{10}Y_{13}^3, Y_2^5 - Y_{13}Y_{14},$ $Y_1Y_2^4 - Y_{12}Y_{14}, Y_1^2Y_2^3 - Y_{11}Y_{14}, Y_{11}^7Y_{12} - Y_{10}Y_{13}^6Y_{14}, Y_{11}^8 - Y_{10}Y_{12}Y_{13}^5Y_{14}$

Type	Equations
(5,6)	$Y_{16}^2 - Y_{15}Y_{17}, Y_{15}Y_{16} - Y_{12}Y_{17}, Y_{13}Y_{16} - Y_{11}Y_{17}, Y_{12}Y_{16} - Y_{10}Y_{17}, Y_{11}Y_{16} - Y_9Y_{17},$ $Y_{10}Y_{16} - Y_8Y_{17}, Y_9Y_{16} - Y_5Y_{17}, Y_6Y_{16} - Y_4Y_{17}, Y_4Y_{16} - Y_3Y_{17}, Y_2Y_{16} - Y_1Y_{17},$ $Y_{15}^2 - Y_{10}Y_{17}, Y_{13}Y_{15} - Y_9Y_{17}, Y_{12}Y_{15} - Y_8Y_{17}, Y_{11}Y_{15} - Y_5Y_{17}, Y_{10}Y_{15} - Y_8Y_{16},$ $Y_9Y_{15} - Y_5Y_{16}, Y_6Y_{15} - Y_3Y_{17}, Y_4Y_{15} - Y_3Y_{16}, Y_2Y_{15} - Y_1Y_{16}, Y_{14}^2 - Y_7Y_{18},$ $Y_7Y_{14} - Y_2Y_{18}, Y_{13}^2 - Y_6Y_{17}, Y_{12}Y_{13} - Y_5Y_{17}, Y_{11}Y_{13} - Y_4Y_{17}, Y_{10}Y_{13} - Y_5Y_{16},$ $Y_9Y_{13} - Y_3Y_{17}, Y_8Y_{13} - Y_5Y_{15}, Y_6Y_{13} - Y_2Y_{17}, Y_5Y_{13} - Y_3Y_{16}, Y_4Y_{13} - Y_1Y_{17},$ $Y_3Y_{13} - Y_1Y_{16}, Y_{12}^2 - Y_8Y_{16}, Y_{11}Y_{12} - Y_5Y_{16}, Y_{10}Y_{12} - Y_8Y_{15}, Y_9Y_{12} - Y_5Y_{15},$ $Y_6Y_{12} - Y_3Y_{16}, Y_4Y_{12} - Y_3Y_{15}, Y_2Y_{12} - Y_1Y_{15}, Y_{11}^2 - Y_3Y_{17}, Y_{10}Y_{11} - Y_5Y_{15},$ $Y_9Y_{11} - Y_3Y_{16}, Y_8Y_{11} - Y_5Y_{12}, Y_6Y_{11} - Y_1Y_{17}, Y_5Y_{11} - Y_3Y_{15}, Y_4Y_{11} - Y_1Y_{16},$ $Y_3Y_{11} - Y_1Y_{15}, Y_2Y_{11} - Y_1Y_{13}, Y_{10}^2 - Y_8Y_{12}, Y_9Y_{10} - Y_5Y_{12}, Y_6Y_{10} - Y_3Y_{15},$ $Y_4Y_{10} - Y_3Y_{12}, Y_2Y_{10} - Y_1Y_{12}, Y_9^2 - Y_3Y_{15}, Y_8Y_9 - Y_5Y_{10}, Y_6Y_9 - Y_1Y_{16},$ $Y_5Y_9 - Y_3Y_{12}, Y_4Y_9 - Y_1Y_{15}, Y_3Y_9 - Y_1Y_{12}, Y_2Y_9 - Y_1Y_{11}, Y_6Y_8 - Y_3Y_{12},$ $Y_4Y_8 - Y_3Y_{10}, Y_2Y_8 - Y_1Y_{10}, Y_7^2 - Y_2Y_{14}, Y_6^2 - Y_2Y_{13}, Y_5Y_6 - Y_1Y_{15}, Y_4Y_6 - Y_1Y_{13},$ $Y_3Y_6 - Y_1Y_{11}, Y_5^2 - Y_3Y_{10}, Y_4Y_5 - Y_1Y_{12}, Y_3Y_5 - Y_1Y_{10}, Y_2Y_5 - Y_1Y_9, Y_4^2 - Y_1Y_{11},$ $Y_3Y_4 - Y_1Y_9, Y_2Y_4 - Y_1Y_6, Y_3^2 - Y_1Y_5, Y_2Y_3 - Y_1Y_4, Y_2Y_{13}Y_{14} - Y_{17}Y_{18},$ $Y_1Y_{13}Y_{14} - Y_{16}Y_{18}, Y_1Y_{11}Y_{14} - Y_{15}Y_{18}, Y_1Y_9Y_{14} - Y_{12}Y_{18}, Y_2Y_6Y_{14} - Y_{13}Y_{18},$ $Y_1Y_6Y_{14} - Y_{11}Y_{18}, Y_1Y_5Y_{14} - Y_{10}Y_{18}, Y_1Y_4Y_{14} - Y_9Y_{18}, Y_1Y_3Y_{14} - Y_5Y_{18},$ $Y_2^2Y_{14} - Y_6Y_{18}, Y_1Y_2Y_{14} - Y_4Y_{18}, Y_1^2Y_{14} - Y_3Y_{18}, Y_2Y_7Y_{13} - Y_{14}Y_{17},$ $Y_1Y_7Y_{13} - Y_{14}Y_{16}, Y_2^2Y_{13} - Y_7Y_{17}, Y_1Y_2Y_{13} - Y_7Y_{16}, Y_1^2Y_{13} - Y_7Y_{15},$ $Y_1Y_7Y_{11} - Y_{14}Y_{15}, Y_1^2Y_{11} - Y_7Y_{12}, Y_1Y_7Y_9 - Y_{12}Y_{14}, Y_1^2Y_9 - Y_7Y_{10},$ $Y_2Y_6Y_7 - Y_{13}Y_{14}, Y_1Y_6Y_7 - Y_{11}Y_{14}, Y_1Y_5Y_7 - Y_{10}Y_{14}, Y_1Y_4Y_7 - Y_9Y_{14},$ $Y_1Y_3Y_7 - Y_5Y_{14}, Y_2^2Y_7 - Y_6Y_{14}, Y_1Y_2Y_7 - Y_4Y_{14}, Y_1^2Y_7 - Y_3Y_{14},$ $Y_2^2Y_6 - Y_7Y_{13}, Y_1Y_2Y_6 - Y_7Y_{11}, Y_1^2Y_6 - Y_7Y_9, Y_1^2Y_5 - Y_7Y_8, Y_1^2Y_4 - Y_5Y_7,$ $Y_2^3 - Y_6Y_7, Y_1Y_2^2 - Y_4Y_7, Y_1^2Y_2 - Y_3Y_7, Y_1^3Y_3 - Y_8Y_{14}, Y_1^5 - Y_8Y_{18}$
(6,6)	$2Y_2^3 - Y_1Y_3^2 + 2Y_1Y_4$
(3,4,4)	$Y_8Y_9 - Y_5Y_{10}, Y_6Y_9 - Y_3Y_{10}, Y_4Y_9 - Y_2Y_{10}, Y_3Y_9 - Y_1Y_{10}, Y_5Y_6 - Y_3Y_8,$ $Y_4Y_6 + Y_4Y_7 - 2Y_1Y_8, Y_4Y_5 - Y_2Y_8, Y_3Y_5 - Y_1Y_8, 2Y_1Y_5 - Y_2Y_6 - Y_2Y_7, Y_3Y_4 - Y_2Y_6,$ $Y_3^2 - Y_1Y_6, Y_2Y_3 - Y_1Y_4, Y_7Y_9^2 - Y_6Y_8Y_{10} + Y_1Y_{10}^2 - Y_1Y_9Y_{11},$ $Y_7^2Y_9 - Y_3Y_6Y_{10} + Y_3Y_7Y_{10} - 2Y_{10}^2 - 2Y_9Y_{11}, Y_5Y_7Y_9 - Y_1Y_9^2 + Y_4Y_7Y_{10} - Y_1Y_8Y_{10},$ $Y_1Y_7Y_9 - 2Y_9^2 - 2Y_8Y_{10}, Y_1^2Y_9 - 2Y_5Y_9 - 2Y_4Y_{10},$ $2Y_7Y_8^2 + 2Y_1Y_9^2 - 2Y_4Y_7Y_{10} - 2Y_1Y_8Y_{10} + Y_2Y_6Y_{11} - Y_2Y_7Y_{11},$ $2Y_6Y_8^2 - 2Y_1Y_9^2 + 2Y_4Y_7Y_{10} - 4Y_1Y_8Y_{10} + Y_2Y_6Y_{11} + Y_2Y_7Y_{11},$ $2Y_5Y_8^2 - 3Y_2Y_8Y_{10} + Y_4^2Y_{11}, Y_7^2Y_8 - 2Y_1Y_7Y_{10} + Y_1Y_3Y_{11} + 4Y_9Y_{10} - 4Y_8Y_{11},$ $Y_6Y_7Y_8 - Y_1Y_6Y_{10} - 2Y_9Y_{10} + 2Y_8Y_{11},$ $2Y_5Y_7Y_8 + 2Y_3Y_8^2 - 3Y_2Y_6Y_{10} - 3Y_2Y_7Y_{10} + 2Y_1Y_4Y_{11},$ $Y_4Y_7Y_8 - 2Y_1Y_8^2 + Y_2Y_7Y_9 + Y_1Y_4Y_{10} - Y_1Y_2Y_{11},$ $Y_3Y_7Y_8 - Y_1Y_3Y_{10} - 2Y_9^2 + 2Y_5Y_{11}, Y_6^2Y_8 - Y_1Y_6Y_{10} - Y_1Y_7Y_{10} + Y_1Y_3Y_{11},$ $Y_3Y_6Y_8 - Y_1Y_3Y_{10} + Y_1^2Y_{11} - 2Y_9^2 - 2Y_8Y_{10},$ $Y_1Y_6Y_8 - Y_1Y_7Y_8 - 2Y_5Y_{10} + 2Y_4Y_{11}, 2Y_5^2Y_8 - 3Y_2Y_5Y_{10} + Y_2Y_4Y_{11},$ $Y_1Y_3Y_8 - 2Y_8^2 - 4Y_5Y_9 + 2Y_2Y_{11}, Y_5Y_7^2 + Y_1^2Y_{11} - 4Y_8Y_{10} - 4Y_5Y_{11},$ $Y_4Y_7^2 - 2Y_1Y_7Y_8 + Y_1^2Y_{10} + 2Y_5Y_{10} - 2Y_4Y_{11},$ $Y_2Y_7^2 - 4Y_8^2 + 2Y_4Y_{10} - 2Y_2Y_{11}, Y_3Y_6Y_7 - 4Y_6Y_{10} - 2Y_7Y_{10} + 2Y_3Y_{11},$ $Y_2Y_6Y_7 - 4Y_5Y_9 - 2Y_4Y_{10} + 2Y_2Y_{11}, Y_1Y_6Y_7 - 2Y_7Y_9 - 4Y_3Y_{10} + 2Y_1Y_{11},$ $2Y_5^2Y_7 + 2Y_1Y_8^2 - 3Y_2Y_7Y_9 - 3Y_1Y_4Y_{10} + 2Y_1Y_2Y_{11},$ $Y_4^2Y_7 + Y_2Y_5Y_7 - Y_1Y_4Y_8 - Y_1Y_2Y_9, Y_1Y_4Y_7 - Y_1^2Y_8 + 2Y_5Y_8 - 2Y_2Y_{10},$ $Y_1Y_3Y_7 - 2Y_6Y_8 - 2Y_1Y_{10}, Y_1Y_2Y_7 - 2Y_4Y_8 - 2Y_2Y_9, Y_1^2Y_7 - 2Y_3Y_8 - 2Y_1Y_9,$ $2Y_6^3 - 3Y_6Y_7^2 - Y_7^3 + 6Y_6Y_{11} + 2Y_7Y_{11}, Y_3Y_6^2 - Y_3Y_7^2 - 2Y_6Y_{10} - 2Y_7Y_{10} + 4Y_3Y_{11},$ $Y_1Y_6^2 - Y_1Y_7^2 - 2Y_7Y_9 - 2Y_3Y_{10} + 4Y_1Y_{11}, Y_1Y_3Y_6 - 4Y_6Y_8 - 2Y_7Y_8 + 2Y_1Y_{10},$ $Y_1Y_2Y_6 - 4Y_5^2 - 2Y_4Y_8 + 2Y_2Y_9, Y_1^2Y_6 - 2Y_5Y_7 - 4Y_3Y_8 + 2Y_1Y_9,$ $2Y_5^3 - 3Y_2Y_5Y_9 + Y_2^2Y_{11}, Y_1^2Y_3 + 2Y_4Y_7 - 6Y_1Y_8, Y_1^2Y_2 - 2Y_4^2 - 2Y_2Y_5,$ $Y_1^3 - 3Y_2Y_6 - Y_2Y_7, Y_9^4 - Y_8^2Y_{10}^2 + Y_4Y_{10}^3 - Y_5Y_9^2Y_{11},$ $2Y_4Y_8^3 - 3Y_4^2Y_8Y_{10} + 2Y_2Y_5^2Y_{11} - Y_2^2Y_9Y_{11}, Y_4^4 - 2Y_2^2Y_5^2 + Y_2^3Y_9$

4.4 Dimension = 4

For dimension equal to 4 we arrive at the borderline of the computational power of nowadays workstations[5] and of the readability of systems of equations because the number of equations exceeds 500. In several cases we were not able to compute the equations of the moduli space, even though we have used a state-of-the-art implementation of the Hilbert-driven Gröbner bases algorithm. Since the order of $G_{f_0}^{\mathbf{w}}$ and E_{f_0} coincides for this case, we omit $|G_{f_0}^{\mathbf{w}}|$. In one case E_{f_0} is generated by diagonal and permutation matrices, which are given in cycle representation. By '-' we denote the fact, that we were unable to compute the corresponding quantity.

Type	μ	UM	$\|E_{f_0}\|$	invars	e-d	nr
(3,16)	30	$xy^{14}, xy^{13}, xy^{12}, xy^{11}$	48	143	-	–
(3,17)	32	$xy^{15}, xy^{14}, xy^{13}, xy^{12}$	51	96	-	–
(3,18)	34	$xy^{16}, xy^{15}, xy^{14}, xy^{13}$	54	31	-	327
(4,9)	24	$x^2y^7, xy^7, x^2y^6, x^2y^5$	36	136	-	-
(4,10)	27	$x^2y^8, xy^8, x^2y^7, x^2y^6$	40	45	10	828
(5,7)	32	$x^3y^5, x^2y^5, x^3y^4, x^3y^3$	35	81	3325	-
(4,4,4)	27	$x^2y^2z^2, xy^2z^2, x^2yz^2, x^2y^2z$	384	5	5	1

Generators of E_{f_0} are given in the table below.

[5] For the calculations we have used a PC with an Athlon 1800XP+ processor with 1GB RAM.

Type	Generators
(3,16)	$\langle\xi_{48}^{10},\xi_{48}^{7},\xi_{48}^{4},\xi_{48}\rangle$
(3,17)	$\langle\xi_{51}^{11},\xi_{51}^{8},\xi_{51}^{5},\xi_{51}^{2}\rangle$
(3,18)	$\langle\xi_3,\xi_3,\xi_3,\xi_3\rangle,\langle\xi_{18}^{2},\xi_{18}^{3},\xi_{18}^{4},\xi_{18}^{5}\rangle$
(4,9)	$\langle\xi_{36}^{22},\xi_{36}^{13},\xi_{36}^{6},\xi_{36}^{26}\rangle$
(4,10)	$\langle-1,i,-1,-1\rangle,\langle\xi_{10}^{2},\xi_{10}^{2},\xi_{10}^{3},\xi_{10}^{4}\rangle$1144 eqns, deg 2 - 10
(5,7)	$\langle\xi_{35}^{1},\xi_{35}^{29},\xi_{35}^{33},\xi_{35}^{30}\rangle$
(4,4,4)	$\langle-1,-1,-1,i\rangle,(23),(234)$

4.5 Dimension = 5

We only were able to compute equations for moduli space of $f_0 = x^4 + y^{12}$ (858 equations) and of $f_0 = x^3 + y^3 + z^3 + w^4$ (81 equations). For all other we provide the embedding dimension and the order of E_{f_0}, which equals the order of $G_{f_0}^{\mathbf{w}}$. In some cases E_{f_0} is generated by diagonal and permutation matrices, which are given in cycle representation.

Type	μ	UM	$\lvert E_{f_0}\rvert$	e-d
(3,19)	36	$xy^{17},xy^{16},xy^{15},xy^{14},xy^{13}$	57	396
(3,20)	38	$xy^{18},xy^{17},xy^{16},xy^{15},xy^{14}$	60	174
(3,21)	40	$xy^{19},xy^{18},xy^{17},xy^{16},xy^{15}$	63	125
(4,11)	30	$x^2y^9,xy^9,x^2y^8,x^2y^7,x^2y^6$	44	183
(4,12)	33	$x^2y^{10},xy^{10},x^2y^9,x^2y^8,x^2y^7$	48	49
(5,8)	28	$x^3y^6,x^2y^6,x^3y^5,x^2y^5,x^3y^4$	40	200
(3,3,7)	24	$xyz^5,yz^5,xz^5,xyz^4,xyz^3$	126	162
(3,4,5)	24	$xy^2z^3,y^2z^3,xyz^3,xy^2z^2,xy^2z$	60	134
(3,3,3,4)	24	$xyzw^2,yzw^2,xzw^2,xyw^2,xyzw$	648	17

Generators of E_{f_0} are given in the table below.

Type	Generators
(3,19)	$\langle\xi_{57}^{13},\xi_{57}^{10},\xi_{57}^{7},\xi_{57}^{4},\xi_{57}\rangle$
(3,20)	$\langle\xi_{60}^{14},\xi_{60}^{11},\xi_{60}^{8},\xi_{60}^{5},\xi_{60}^{2}\rangle$
(3,21)	$\langle\xi_3,\xi_3,\xi_3,\xi_3,\xi_3\rangle,\langle\xi_{21}^{2},\xi_{21}^{3},\xi_{21}^{4},\xi_{21}^{5},\xi_{21}^{6}\rangle$
(4,11)	$\langle\xi_{44}^{14},\xi_{44}^{3},\xi_{44}^{10},\xi_{44}^{6},\xi_{44}^{2}\rangle$
(4,12)	$\langle-1,i,-1,-1,-1\rangle,\langle\xi_{12}^{2},\xi_{12}^{2},\xi_{12}^{3},\xi_{12}^{4},\xi_{12}^{5}\rangle$
(5,8)	$\langle\xi_{40}^{6},\xi_{40}^{14},\xi_{40},\xi_{40}^{9},\xi_{40}^{36}\rangle$
(3,3,7)	$\langle\xi_{21},\xi_{21}^{15},\xi_{21},\xi_{21}^{19},\xi_{21}^{16}\rangle,(23)$
(3,4,5)	$\langle\xi_{60}^{2},\xi_{60}^{2},\xi_{60}^{17},\xi_{60}^{38},\xi_{60}^{14}\rangle$
(3,3,3,4)	$\langle\xi_{12}^{10},-1,\xi_{12}^{10},\xi_{12}^{10},\xi_{12}^{7}\rangle,(23),(234)$

5 Conclusion

We have presented equations for moduli spaces of semi Brieskorn-Pham singularities w.r.t. right equivalence of dimension $1,2,3$ (with one exception) and generators of E_{f_0} together with the embedding dimension for dimension $1,2,3,4,5$. From dimension 2 on, there is no hope for finding a unique normal form because of the complexity of the corresponding moduli spaces. Indeed, if we drop the requirement for uniqueness, we may consider the functions $F(\mathbf{x},t) = f_0 + \sum_{m_i\in B_-} t_i m_i$ as normal forms with finite ambiguity of $\lvert E_{f_0}\rvert$.

If one considers the coarser relation contact equivalence instead of right equivalence, coarse moduli spaces for semiquasihomogeneous singularities still exist, but one has to fix more invariants and the construction is much more involved. In particular, the corresponding groups need not be reductive! We refer to [GHP 1997] for the existence proof. In this case, not even a normal form of finite ambiguity may be found.

References

[Arnold 1972] Arnold, V.I. *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities.* Functional Anal. Appl. 6 (1972) 254 - 272.

[Arnold 1974] Arnold, V.I. *Normal forms of functions in neighborhoods of degenerate critical points.* Usp. Mat. Nauk.29, No. 2, 1974.

[AVG 1985] Arnold, V.I, Varchenko, A.N., Gusein-Zade, S.M. *Singularities of differentiable mappings Vol. I,II.* Birkhäuser, Basel 1985.

[Bayer 2000] Bayer, T. `qhmoduli.lib`. A SINGULAR 2.0 library for computing moduli spaces of semiquasihomogeneous singularities with fixed principal part. (2000).

[Bayer 2002] Bayer, T. *Algorithms for the computation of moduli spaces for semiquasihomogeneous singularities.* To appear in: Revista Mathematica Complutense.

[Derksen 1997] Derksen, H.,*Computation of Invaraiants for Reductive Groups.* Adv. Math. **141** (1999) 366 - 384.

[GHP 1997] Greuel, G.-M., Hertling, K., Pfister, G. *Moduli spaces of semiquasihomogeneous singularities with fixed principal part.* J. Algebraic Geometry 6 (1997) 169 - 199.

[GPS01] Greuel, G.-M., Pfister, G., and Schönemann, H. SINGULAR 2.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2001). `http://www.singular.uni-kl.de`.

[Heydtmann 1997] Heydtmann, A. E., `finvar.lib`. A SINGULAR 2.0 library for computing invariant rings of finite groups (1997).

[HM 1998] Harris, J., Morrison, I. *Moduli of Curves* Graduate Texts in Mathematics 187, Springer Verlag New York Berlin Heidelberg 1998.

[Kraft 1985] Kraft, H. *Geometrische Methoden in der Invariantentheorie.* 2. Auflage, Aspekte der Mathematik D1, Vieweg Verlag, Braunschweig 1985.

[Saito 1974] Saito, K. *Einfach-elliptische Singularitäten.* Invent. Math. 23 (1974) 289 - 325.

[SAF 1986] J. Estrada Sarlabous,J. Arocha, A. Fuentes. *Classification of non–degenerate quasihomogeneous functions with inner modality 6.* An. Inst. Mat. Univ. Nac. Antónoma Mexico 26 (1986) 21–39.

[Traverso 1996] Traverso, C. *Hilbert Functions and the Buchberger Algorithm.* J. Symb. Comp. Vol. 22 No. 4 (1996) 355 - 376.

[YS 1979] Yoshinaga, E., Suzuki, M. *Normal Forms of Non-Degnerate Quasihomogeneous Functions with Inner Modality* ≤ 4. Invent. Math. 55 (1979) 185 - 206.