Computations in Modules over Commutative Domains

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Часть I

System of Linear Equations

Let R be a commutative domain, F be the field of fractions of R. $A \in R^{n \times m}, c \in R^n, n \leq m, A^\star = (A, c) = (a_{ij})$ and,

$$Ax = c$$

be a system of linear equations over R

$$x_i = (\delta^n_{i,m+1} - \sum_{j=n+1}^{m} x_j \delta^n_{ij})(\delta^n)^{-1},\ i = 1 \dots n,$$

where $x_j, j = n+1, \dots, m$, are the free variables, and the determinant $\delta^n \neq 0$.

We denote by $\delta^k, k = 1, \ldots, n$ the left upper corner minor of matrix A of order k, and by δ^k_{ij} the corner minor of matrix A where columns i and j have been interchanged. We assume that all corner minors $\delta^k, k = 1, \ldots, n$ are different from 0.

The determinant identity

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & b \\ d & e \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} e & f \\ h & k \end{vmatrix} \end{vmatrix} \cdot e^{-1}$$

or in the more general form

$$\hat{a}_{ij}^{k+1} = \begin{vmatrix} \hat{a}_{i-1,j-1}^{k} & \hat{a}_{i-1,j}^{k} \\ \hat{a}_{i,j-1}^{k} & \hat{a}_{ij}^{k} \end{vmatrix} \cdot (\hat{a}_{i-1,j-1}^{k-1})^{-1}$$

where

$$\hat{a}_{ij}^{k+1} = |A_{j-k,\dots,j:(columns)}^{i-k,\dots,i:(rows)}|$$

$$a_{ij}^{k+1} = (a_{kk}^{k} a_{ij}^{k} - a_{ik}^{k} a_{kj}^{k})(a_{k-1,k-1}^{k-1})^{-1},$$
$$k = 2, \ldots, n-1,\ i = k+1, \ldots, n,\ j = k+1, \ldots, m,$$

where

$$a_{ij}^{k+1} = |A_{1,\ldots,k,j:(columns)}^{1,\ldots,k,i:(rows)}|$$

$$\begin{pmatrix}
a^1_{1,1} & a^1_{1,2} & \cdots & a^1_{1,n-1} & a^1_{1,n} & a^1_{1,n} & \cdots & a^1_{1,m+1} \\
0 & a^2_{2,2} & \cdots & a^2_{2,n-1} & a^2_{2,n} & a^2_{2,n+1} & \cdots & a^2_{2,m+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a^{n-1}_{n-1,n-1} & a^{n-1}_{n-1,n} & a^{n-1}_{n-1,n+1} & \cdots & a^{n-1}_{n-1,m+1} \\
0 & 0 & \cdots & 0 & a^n_{n,n} & a^n_{n,n+1} & \cdots & a^n_{n,m+1}
\end{pmatrix}$$

$a^k_{k,k} \neq 0,\ k = 1, \ldots, n-1$

δ^n_{ij}. δ^k_{ij} is the corner minor of order k of the matrix A after column i has been interchanged with column j. The determinant identity of the backward direction algorithm is:

$$\delta^n_{ij} = \Big(a^n_{nn} a^i_{ij} - \sum_{k=i+1}^{n} a^i_{ik} \delta^n_{kj}\Big)(a^i_{ii})^{-1},\ \ i = n-1, \ldots 1,\ \ j = n+1, \ldots, m.$$

$$\begin{pmatrix} a^n_{n,n} & 0 & \cdots & 0 & 0 & \delta^n_{1,n} & \cdots & \delta^n_{1,m+1} \\ 0 & a^n_{n,n} & \cdots & 0 & 0 & \delta^n_{2,n+1} & \cdots & \delta^n_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^n_{n,n} & 0 & \delta^n_{n-1,n+1} & \cdots & \delta^n_{n-1,m+1} \\ 0 & 0 & \cdots & 0 & a^n_{n,n} & \delta^n_{n,n+1} & \cdots & \delta^n_{n,m+1} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 3 & 6 & 0 & -6 \\ 0 & -1 & -1 & 7 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & -2 & 13 & -9 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix} \rightarrow \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix}$$

The number of operations, necessary for the procedure of forward and backward direction, is
$N^m = (9n^2m - 5n^3 - 3nm - 3n^2 - 6m + 8n)/6$,
$N^d = (3n^2m - n^3 - 3nm - 6n^2 + 13n - 6)/6$
$N^a = (6n^2m - 4n^3 - 6nm + 3n^2 + n)/6$.

$$\delta^{k+1}_{k+1,j} = a_{k+1,k+1}\delta^k_{kk} - \sum_{p=1}^{k} a_{k+1,p}\delta^k_{pj}, j = k+1\dots m,$$

$$\delta^{k+1}_{ij} = (\delta^{k+1}_{k+1,k+1}\delta^k_{i,j} - \delta^{k+1}_{k+1,j}\delta^k_{i,k+1})/\delta^k_{k,k},$$
$$k = 1,\dots,n-1,\ i = 1,\dots,k,\ j = k+2,\dots,m.$$

$$\begin{pmatrix}
a^k_{k,k} & 0 & \cdots & 0 & \delta^k_{1,k+1} & \cdots & \delta^k_{1,m+1} \\
0 & a^k_{k,k} & \cdots & 0 & \delta^k_{2,k+1} & \cdots & \delta^k_{2,m+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a^k_{k,k} & \delta^k_{k,k+1} & \cdots & \delta^k_{k,m+1} \\
a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,m+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,k} & a_{n,k+1} & \cdots & a_{n,m+1}
\end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 11 & 0 & 0 & -5 & 16 \\ 0 & 11 & 0 & 8 & 14 \\ 0 & 0 & 11 & -4 & -18 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 11 & 0 & 0 & -5 & 16 \\ 0 & 11 & 0 & 8 & 14 \\ 0 & 0 & 11 & -4 & -18 \end{pmatrix} \rightarrow \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix}$$

$$N^m = (9n^2m - 6n^3 - 3nm - 6m + 6n)/6,$$
$$N^d = (3n^2m - 2n^3 - 3nm - 6m + 2n + 12)/6$$
$$N^a = (6n^2m - 4n^3 - 6nm + 3n^2 + n)/6.$$

Number of Operations			
Method	Multiplications	Divisions	Add./Substr.
FB	$\frac{(4n^3+3n^2-n-6)}{6}$	$\frac{(2n^3-6n^2+10n-6)}{6}$	$\frac{(2n^3+3n^2-5n)}{6}$
OP	$\frac{(n^3+2n^2-n-2)}{2}$	$\frac{(n^3-7n+6)}{6}$	$\frac{(2n^3+3n^2-5n)}{6}$

$$A_{k,c}^{r,l,(p)} = \begin{pmatrix} a^p_{r+1,k+1} & a^p_{r+1,k+2} & \cdots & a^p_{r+1,c} \\ a^p_{r+2,k+1} & a^p_{r+2,k+2} & \cdots & a^p_{r+2,c} \\ \vdots & \vdots & \ddots & \vdots \\ a^p_{l,k+1} & a^p_{l,k+2} & \cdots & a^p_{l,c} \end{pmatrix},$$

$$G_{k,c}^{r,l,(p)} = \begin{pmatrix} \delta^p_{r+1,k+1} & \delta^p_{r+1,k+2} & \cdots & \delta^p_{r+1,c} \\ \delta^p_{r+2,k+1} & \delta^p_{r+2,k+2} & \cdots & \delta^p_{r+2,c} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^p_{l,k+1} & \delta^p_{l,k+2} & \cdots & \delta^p_{l,c} \end{pmatrix},$$

$G^{r,l,(p)}_{k,c}, A^{r,l,(p)}_{k,c} \in \mathbf{R}^{(l-r)\times(c-k)}$, $0 \leq k < n$, $k < c \leq n$, $0 \leq r < m$, $r < l \leq m$, $1 \leq p \leq n$.
We describe one recursive step reducing the matrix $\tilde{A} = A^{k,l,(k+1)}_{k,c}$ to the diagonal form

$$\tilde{A} \rightarrow (\delta^l I_{l-k}, \hat{G})$$

where

$$\tilde{A} = A^{k,l,(k+1)}_{k,c}, \quad \hat{G} = G^{k,l,(l)}_{l,c}$$

$0 \leq k < c \leq m$, $k < l \leq n$, $l < c$. Note that if $k = 0$, $l = n$ and $c = m$, then we obtain the solution of the original system.

Description of One Step of the Recursive Method

$$\tilde{A} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \rightarrow_1 \begin{pmatrix} \delta^s I_{s-k} & G_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} \rightarrow_2 \begin{pmatrix} \delta^s I_{s-k} & G_2^1 \\ 0 & \hat{A}_2^2 \end{pmatrix} \rightarrow_3$$

$$\rightarrow_3 \begin{pmatrix} \delta^s I_{s-k} & G_{2'}^1 & G_{2''}^1 \\ 0 & \delta^l I_{l-s} & \hat{G}_{2''}^2 \end{pmatrix} \rightarrow_4 \begin{pmatrix} \delta^l I_{s-k} & 0 & \hat{G}_{2''}^1 \\ 0 & \delta^l I_{l-s} & \hat{G}_{2''}^2 \end{pmatrix} =$$

$$= \begin{pmatrix} \delta^l I_{l-k} & \hat{G} \end{pmatrix}$$

s: $k < s < l$ $\tilde{A}$:

$$\tilde{A} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix},$$

where $A^1 = A_{k,c}^{k,s,(k+1)}$ is the upper part of the matrix $\tilde{A}$ consisting of $s-k$ rows and $A^2 = A_{k,c}^{s,l,(k+1)}$ is the lower part of the matrix $\tilde{A}$.

$$A^1 \rightarrow (\delta^s I_{s-k}, G_2^1), \qquad (I)$$

where $A^1 \in \mathbf{R}^{(s-k)\times(c-k)}$, $G_2^1 = G_{s,c}^{k,s,(s)}$.

Let $A^2 = (A_1^2, A_2^2)$ where $A_1^2 = A_{k,s}^{s,l,(k+1)}$ and $A_2^2 = A_{s,c}^{s,l,(k+1)}$ consisting of $s-k$ and $c-s$ columns respectively, $\delta^k \neq 0$. The matrix $\hat{A}_2^2 = A_{s,c}^{s,l,(s+1)}$ is computed with the help of the matrix identity

$$\hat{A}_2^2 = (\delta^s \cdot A_2^2 - A_1^2 \cdot G_2^1)(\delta^k)^{-1}. \qquad (II)$$

$$\hat{A}_2^2 \rightarrow (\delta^l I_{l-s}, \hat{G}_{2''}^2), \qquad (III)$$

where $\hat{A}_2^2 \in \mathbf{R}^{(l-s)\times(c-s)}$ and $\hat{G}_{2''}^2 = G_{l,c}^{s,l,(l)}$.
Let $G_2^1 = (G_{2'}^1, G_{2''}^1)$, where the blocks $G_{2'}^1 = G_{s,l}^{k,s,(s)}$ and $G_{2''}^1 = G_{l,c}^{k,s,(s)}$ contain $l-s$ and $c-l$ columns respectively, and $\delta^s \neq 0$.

The matrix $\hat{G}^1_{2''} = G^{k,s,(l)}_{l,c}$ is computed with the help of the matrix identity

$$\hat{G}^1_{2''} = (\delta^l \cdot G^1_{2''} - G^1_{2'} \cdot \hat{G}^2_{2''})(\delta^s)^{-1}. \qquad (IV)$$

In the result we obtain δ^l and

$$\hat{G} = \begin{pmatrix} \hat{G}^1_{2''} \\ \hat{G}^2_{2''} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \\ -1 \end{pmatrix} A^1 = A_{05}^{02(1)} \longrightarrow (\delta^2 I_2,\ \ G_{25}^{02(2)}) \qquad (I)$$

$$A_{05}^{01(1)} \longrightarrow (\delta^1 I_1,\ \ G_{15}^{01(1)}) = (3; 1, 1, -1, 4) \qquad (i)$$

$$\delta^0 A_{15}^{12(2)} = \delta^1 A_{15}^{12(1)} - A_{01}^{12(1)} G_{15}^{01(1)} =$$

$$= 3(2,0,1,4) - (1)(1,1,-1,4) = (5,-1,4,8), \quad \delta^0 \equiv 1. \qquad (ii)$$

$$A_{15}^{12(2)} \rightarrow (\delta^2 I_1, \;\; G_{25}^{12(2)}) = (5;-1,4,8) \qquad (iii)$$

$$\delta^1 G_{25}^{01(2)} = \delta^2 G_{25}^{01(1)} - G_{12}^{01(1)} G_{25}^{12(2)} =$$

$$= 5(1,-1,4) - (1)(-1,4,8) = (6,-9,12) \qquad (iv)$$

$$G_{25}^{01(2)} = (2, -3, 4)$$

$$(\delta^2 I_2,\ \ G_{25}^{02(2)}) = \left(\begin{array}{cccccc} 5 & 0; & 2 & -3 & 4 \\ 0 & 5; & -1 & 4 & 8 \end{array} \right)$$

$$\hat{A}_2^2 = A_{25}^{24(3)} \qquad (II)$$

$$\delta^0 A_{25}^{24(3)} = \delta^2 A_{25}^{24(1)} - A_{02}^{24(1)} G_{25}^{02(2)} =$$

$$= 5 \cdot \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 & 4 \\ -1 & 4 & 8 \end{pmatrix} =$$

$$= \begin{pmatrix} 11 & -4 & -18 \\ -2 & 13 & -9 \end{pmatrix}$$

$$\delta^0 \equiv 1; \quad A_{25}^{24(3)} = \begin{pmatrix} 11 & -4 & -18 \\ -2 & 13 & -9 \end{pmatrix}$$

$$A_{25}^{24(3)} \rightarrow (\delta^4 I_2 \;\; G_{45}^{24(4)}) \qquad (III)$$

$$A_{25}^{23(3)} \rightarrow (\delta^3 I_1, \;\; G_{35}^{23(3)}) = (11; -4, -18) \qquad (i)$$

$$\delta^2 A_{35}^{34(4)} = \delta^3 A_{35}^{34(4)} - A_{23}^{34(3)} G_{35}^{23(3)} =$$

$$= 11(13, -9) - (-2)(-4, -18) = (135, -135)$$

$$A_{35}^{34(4)} = (27, -27) \qquad (ii)$$

$$A_{35}^{34(4)} \longrightarrow (\delta^4 I_1,\ \ G_{45}^{34(4)}) = (27, -27) \qquad (iii)$$

$$\delta^3 G_{45}^{23(4)} = \delta^4 G_{45}^{23(3)} - G_{34}^{23(3)} G_{45}^{34(4)} =$$

$$= 27(-18) - (-4)(-27) = -594, \quad G_{45}^{23(4)} = (-54) \qquad (iv)$$

$$(\delta^4 I_2,\ \ G_{45}^{24(4)} = \left(\begin{array}{ccc} 27 & 0; & -54 \\ 0 & 27; & -27 \end{array} \right)$$

$$\hat{G}_{2''}^{1} = G_{45}^{02(4)} \qquad (V)$$

$$\delta^2 G_{45}^{02(4)} = \delta^4 G_{45}^{02(2)} - G_{24}^{02(2)} G_{45}^{24(4)} =$$

$$= 27 \begin{pmatrix} 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -54 \\ -27 \end{pmatrix} = \begin{pmatrix} 135 \\ 270 \end{pmatrix}$$

$$G_{45}^{02(4)} = \begin{pmatrix} 27 \\ 54 \end{pmatrix}$$

$$\delta^4 = 27; \quad G_{45}^{04(4)} = \begin{pmatrix} 27 \\ 54 \\ -54 \\ -27 \end{pmatrix}$$

Example: Description of the First Step

$$A = \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} A \rightarrow_1$$

$$\begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow_2 \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & -2 & 13 & -9 \end{pmatrix} \rightarrow_3$$

$$\rightarrow_3 \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow_4 \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix}$$

Complexity of the Recursive Method is $O(mn^{\beta-1})$.
We can obtain an exact estimate. For $n = 2^N$, $m = n + 1$ and $\beta = \log_2 7$ the number of multiplication operations is

$$\frac{7}{15} n^{\log_2 7} + n^2(\log_2 n - \frac{2}{3}) + n(2 \log_2 n + \frac{1}{5}).$$

For $n = 2^N$, $\beta = 3$ the number of multiplications and divisions is
$N^m = (6n^2 m - 4n^3 + (6nm - 3n^2) \log_2 n - 6nm + 4n)/6$,
$N^d = ((6nm - 3n^2) \log_2 n - 6nm - n^2 + 6m + 3n - 2)/6$.
The number of multiplication operations for $m = n + 1$ is $(1/3)n^3 + O(n^2)$. The estimations for the previous two methods are, respectively, $n^3 + O(n^2)$ and $(2/3)n^3 + O(n^2)$.

Часть II

Adjoint Matrix

The best method for computing the matrix determinant and adjoint matrix in the arbitrary commutative ring was suggested in the papers by Kaltofen [9] and Kaltofen and Villard [10]. Its complexity is $O(n^{\beta+1/3}\log n \log\log n)$.

Let $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ be an invertible matrix and A an invertible block.

Then

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (D - BA^{-1}C)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} A^{-1} \\ 0 \end{pmatrix}$$

In case $n = 2^p$ it will take 2^{p-1} inversions of 2×2 blocks and 2^{p-k} multiplications of $2^k \times 2^k$ blocks.

$$n^{log 7} - n/2$$

$$O(n^{\beta})$$

,

$$O(n^{\beta})$$

.

Let R be a commutative ring, and let $\mathcal{A} = (a_{i,j})$ be a square matrix of order n over the ring R.

$$\mathcal{A}_t^{(s)} = (a_{i,j}^s)_{j=s,\dots,t}^{i=s,\dots,t} \quad \text{and} \quad \mathcal{G}_s^{(t)} = (\delta_{t(i,j)})_{j=t+1,\dots,n}^{i=s,\dots,t}$$

Theorem

Let $\mathcal{A}$ be a square block matrix of order n over the ring R; that is,

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where A is a square block of order s, $(1 < s < n)$, the determinant of which, δ_s, is neither zero nor a zero divider in R. Then, the adjoint matrix $\mathcal{A}^*$ can be written as the product

$$\mathcal{A}^* = \begin{pmatrix} \delta_s^{-1}\delta_n I & -\delta_s^{-1}FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_s I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}, \qquad (*)$$

where $F = A^*$, $G = \delta_s^{-n+s+1}\mathcal{A}_n^{(s+1)*}$, I is the identity matrix and we have the identity

$$\mathcal{A}_n^{(s+1)} = \delta_s D - BFC.$$

Theorem

Let $\mathcal{A}_n^{(s+1)}$ be a square block matrix of order $n-s$, $(s>0,\ n-s>2)$, over the ring R; that is,

$$\mathcal{A}_n^{(s+1)} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{pmatrix},$$

where $\mathbf{A}$ is a square block of order $t-s$, $(1<s<t<n)$, and δ_s and δ_t are neither zero nor zero dividers in R. Then, the matrix $\delta_s^{-n+s+1}\mathcal{A}_n^{(s+1)*}$ can be written as the product

$$\begin{pmatrix} \delta_t^{-1}\delta_n I & -\delta_t^{-1}\mathbf{FC} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \delta_s^{-1}\mathbf{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathbf{B} & \delta_t I \end{pmatrix} \begin{pmatrix} \mathbf{F} & 0 \\ 0 & I \end{pmatrix}, \quad (**)$$

where $\mathbf{F} = \delta_s^{-t+s+1}\mathcal{A}_t^{(s+1)*}$, $\mathbf{G} = \delta_t^{-n+t+1}\mathcal{A}_n^{(t+1)*}$, I is the identity matrix and we have the identity

$$\mathcal{A}_n^{(t+1)} = \delta_s^{-1}(\delta_t\mathbf{D} - \mathbf{BFC}).$$

Remark 1: If $n = s + 2$, then, $\mathcal{A}_n^{(s+1)*} = \begin{pmatrix} a_{n,n}^{s+1} & -a_{n-1,n}^{s+1} \\ -a_{n,n-1}^{s+1} & a_{n-1,n-1}^{s+1} \end{pmatrix}$.

And if $n = s + 1$, then $\mathcal{A}_n^{(s+1)*} = 1$.

Remark 2: In the factorization $(*)$ δ_n can be found from Sylvester's identity $\delta_n = \delta_s^{-n+s+1} \det \mathcal{A}_n^{(s+1)}$. And if $n = s + 2$, then $\delta_n = \det \mathbf{G}$. In an analogous way we can find δ_t and δ_n, in the factorization $(**)$:

The dimensions of the upper left block A (of the initial square block matrix $\mathcal{A}$) may be chosen arbitrarily. The case will be examined when the dimensions of block A are powers of two. Let n be the order of the matrix $\mathcal{A}$, $2^h < n \leq 2^{h+1}$ and assume that all minors δ_{2^i}, $i = 1, 2, \dots$ are not zero or zero dividers of the ring R. According to Theorems 1 and 2 we are going to sequentially compute adjoint matrices for the upper left blocks of order $2, 4, 8, 16, \dots$ of matrix $\mathcal{A}$.

1. For the block of order 2 we have:

$$A^2_{2,2} = (a_{i,j})_{i,j=1,2},\ \delta_2 = \det A^1_{2,2},$$

$$A^{2*}_{2,2} = \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}.$$

2. For the block of order 4 we have:

$$A^{4*}_{4,4} = \begin{pmatrix} \delta_2^{-1}\delta_4 I & -\delta_2^{-1}FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_2 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix},$$

$F = A^{2*}_{2,2},\ B = (a_{i,j})^{i=3,4}_{j=1,2},$
$C = (a_{i,j})^{i=1,2}_{j=3,3},\ D = (a_{i,j})_{i,j=3,4},\ \mathcal{A}^{(3)}_4 = \delta_2 D - BFC = (a^3_{i,j})_{i,j=3,4},$
$G = \delta_2^{-1}\mathcal{A}^{(3)*}_4,\ \delta_4 = \det G.$

3. For the block of order 8 we have:

$$A_{8,8}^{8*} = \begin{pmatrix} \delta_4^{-1}\delta_8 I & -\delta_4^{-1}FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_4 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix},$$

$F = A_{4,4}^{4*}$, $B = (a_{i,j})_{j=1,\dots,4}^{i=5,\dots,8}$,
$C = (a_{i,j})_{j=5,\dots,8}^{i=1,\dots,4}$,
$D = (a_{i,j})_{i,j=5,\dots,8}$,

$$G = \begin{pmatrix} \delta_6^{-1}\delta_8 I & -\delta_6^{-1}\mathbf{FC} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \delta_4^{-1}\mathbf{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathbf{B} & \delta_6 I \end{pmatrix} \begin{pmatrix} \mathbf{F} & 0 \\ 0 & I \end{pmatrix},$$

$\mathcal{A}_8^{(5)} = \delta_4 D - BFC = (a^5_{i,j})_{i,j=5,\dots,8}$,
$\mathbf{F} = \delta_4^{-3}\mathcal{A}_6^{(5)*}$,
$\delta_6 = \det \mathbf{F}$, $\mathbf{B} = (a^5_{i,j})^{i=7,8}_{j=5,6}$,
$\mathbf{C} = (a^5_{i,j})^{i=5,6}_{j=7,8}$,
$\mathbf{D} = (a^5_{i,j})_{i,j=7,8}$, $\mathcal{A}_8^{(7)} = \delta_4^{-1}(\delta_6\mathbf{D} - \mathbf{BFC}) = (a^7_{i,j})_{i,j=7,8}$,
$\mathbf{G} = \delta_6^{-1}\mathcal{A}_8^{(7)*}$,
$\delta_8 = \det \mathbf{G}$.

Example:

$$A = \begin{pmatrix} 0 & 2 & -2 & 2 \\ 1 & -3 & 1 & -2 \\ 3 & 0 & -3 & 0 \\ -1 & 3 & -1 & 1 \end{pmatrix}.$$

1. For the second order block we have:

$$A_{0,2}^{0,2} = \begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix}, \ \alpha^2 = \det A_{0,2}^{0,2} = -2, \ A_{0,2}^{0,2*} = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}.$$

2. For the fourth order block we have:

$$F = A_{0,2}^{0,2*},\ B = \begin{pmatrix} 3 & 0 \\ -1 & 3 \end{pmatrix},\ C = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix},\ D = \begin{pmatrix} -3 & 0 \\ -1 & 1 \end{pmatrix},$$

$$A_{2,4}^{3;2,4} = \alpha^2 D - BFC = \begin{pmatrix} -6 & 6 \\ 0 & 2 \end{pmatrix},\ G = (\alpha^2)^{-1} A_{2,4}^{3;2,4*} = \begin{pmatrix} -1 & 3 \\ 0 & 3 \end{pmatrix},$$

$$\alpha^4 = (\alpha^2)^{-1} \det A_{2,4}^{3;2,4} = 6,$$

$$A^* = \begin{pmatrix} (\alpha^2)^{-1}\alpha^4 I & -(\alpha^2)^{-1} FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \alpha^2 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 & 2 & -1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & -2 & 0 \\ 1 & -3 & 0 & -2 \end{pmatrix} \begin{pmatrix} -3 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & -12 & 4 & -6 \\ -6 & -6 & 2 & 0 \\ -9 & -12 & 2 & -6 \\ 0 & -6 & 0 & -6 \end{pmatrix}.$$

Complexity Estimation

Let $\gamma n^{\beta} + o(n^{\beta})$ be an asymptotic estimation of the number of operations for multiplying two matrices of order n. Then the complexity of computing the adjoint matrix of order $n = 2^p$ is

$$F(n) = 6\gamma n^{\beta} \frac{1 - (n/2)^{1-\beta}}{2^{\beta} - 2} + o(n^{\beta})$$

Часть III

Characteristic polynomial

In the case of an arbitrary commutative ring, the best algorithms for computing the characteristic polynomial are Chistov's algorithm [6] and the improved Berkowitz algorithm [3]. The complexity of these methods is $O(n^{\beta+1} \log n)$. We present the best method to date — for computations in commutative domains — which has complexity $O(n^3)$.

Let $A = (a_{ij})$ be an $n \times n$ matrix over the ring R. If all the diagonal minors δ_k $(k = 1, \ldots, n-1)$ of matrix A are *not* zero, then the following identity holds

$$A_u = \tilde{L}A,$$

where A_u is an upper triangular matrix and $\tilde{L}$ is a lower triangular matrix with determinant different from zero, such that

$$\tilde{L} = \mathbf{D}_{n-2}^{-1}\tilde{\mathbf{L}}_{n-1}\cdots\mathbf{D}_{1}^{-1}\tilde{\mathbf{L}}_2\tilde{\mathbf{L}}_1$$

$\tilde{\mathbf{L}}_k = \mathrm{diag}(I_{k-1}, \tilde{L}_k)$, $\mathbf{D}_k = \mathrm{diag}(I_k, D_k)$, where I_k is the identity matrix of order k, $D_k = \delta_k I_{n-k}$,

$$L_k = \begin{pmatrix} \delta_k & 0 \\ v_k & I_{n-k} \end{pmatrix}, \tilde{L}_k = \begin{pmatrix} 1 & 0 \\ -v_k & \delta_k I_{n-k} \end{pmatrix}$$

$v_k = (a^k_{k+1,k}, \dots, a^k_{n,k})^T$, $A_u = (a^{(n)}_{i,j})$ is an $n \times n$ matrix, and $a^{(n)}_{i,j} = a^i_{i,j}$, for $i \leq j$, $a^{(n)}_{i,j} = 0$, for $i > j$.
The proof is based on Sylvester's identity

$$a^{k-1}_{k-1,k-1} a^{k+1}_{i,j} = a^k_{k,k} a^k_{i,j} - a^k_{i,k} a^k_{k,j}.$$

The factorization of matrix A into upper and lower triangular matrices is the result of the forward direction part — of the forward and backward direction algorithm.

Let $A_u^{(k)} = (a_{i,j}^{(k)})$ be an $n \times n$ matrix, $k = 1, \ldots, n$, with $a_{i,j}^{(k)} = a_{i,j}^{i}$ for $i \leq j < k$, $a_{i,j}^{(k)} = a_{i,j}^{k}$, $i \geq k$, $j \geq k$, and the remaining elements zero. Then $A_u = \tilde{L}A$ reduces to the identities

$$A_u^{(2)} = \tilde{\mathbf{L}}_1 A; \quad A_u^{(k+1)} = \mathbf{D}_{k-1}^{-1} \tilde{\mathbf{L}}_k A_u^{(k)}, \quad k = 2, \ldots, n-1,$$

which subsequently enable the computation of matrices $A_u^{(k)}$, $k = 2, 3, \ldots, n$, such that all the elements of the matrices $\mathbf{D}_k$ and $\tilde{\mathbf{L}}_k$ are elements of the matrix $A_u^{(k)}$.

The requirement that the diagonal minors δ_k $(k = 1, 2, \ldots, n-1)$ be different from zero may be weakened. If a diagonal minor δ_k of order k is equal to zero, and in column v_k there is a nonzero element $a^k_{i,k}$, then rows i and k must be interchanged; that is, multiply on the left the matrix of interchanges $P_k = P_{(i,k)} = I_n + E_{ik} + E_{ki} - E_{kk} - E_{ii}$, where E_{ik} denotes a matrix in which all elements are zero except element (i, k), which is equal to one.

And if $\delta_k = 0$ and $v_k = 0$, then necessarily $P_k = \tilde{\mathbf{L}}_k = \mathbf{D}_{k-1} = I_n$, $\mathbf{D}_k = \mathbf{D}_{k-1}$.
The factorization formula remains as before, only now

$$\tilde{\mathbf{L}}_k = \mathrm{diag}(I_{k-1}, \tilde{L}_k) P_k.$$

Note the following identities, which will be subsequently needed:

$$\tilde{L}_k L_k = D_k, \quad \tilde{L} L = T,$$

where

$$L = \mathbf{L}_1 \mathbf{L}_2 \cdots \mathbf{L}_{n-1},$$

$\mathbf{L}_k = P_k^{-1} \mathrm{diag}(I_{k-1}, L_k)$, T is a diagonal matrix defined by, $T = S_1 S_2$, where $S_1 = \mathrm{diag}(1, S)$, and $S_2 = \mathrm{diag}(S, 1)$, with $S = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_{n-1})$.
To indicate the matrix A from which a given triangular or diagonal matrix was computed, we write $L = L(A)$, $T = T(A)$.

Let $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ be a matrix over R with blocks $\mathbf{a}$ of order $p \times p$ and $\mathbf{d}$ of order $n \times n$. We will call matrix A upper p-triangular, if the block $(\mathbf{c}, \mathbf{d})$ looks like an upper triangular matrix.
We will denote with calligraphic letters block-diagonal matrices of order $(n+p) \times (n+p)$ of the type $\mathrm{diag}(I_p, G) = \mathcal{G}$, where G is a $p \times p$ matrix.

Let G be some $p \times p$ matrix and let $\tilde{L} = \tilde{L}((\mathbf{c}, \mathbf{d})\mathcal{G})$, and $T = T(((\mathbf{c}, \mathbf{d})\mathcal{G}))$. If we take now $G = L$, $\tilde{\mathcal{L}} = \mathrm{diag}(I_p, \tilde{L})$, $\mathcal{L} = \mathrm{diag}(I_p, L)$, then the matrix

$$A_u = \tilde{\mathcal{L}} A \mathcal{L}$$

will become an upper p-triangular matrix, and matrix $T^{-1}A_u$ will be similar to A.

The cofactors L and $\tilde{L}$ of the matrix can be computed sequentially. Since $((\mathbf{c}, \mathbf{d})\mathcal{G}) = (\mathbf{c}, \mathbf{d}L)$ and the first p of the columns of the matrix $(\mathbf{c}, \mathbf{d}L)$ constitute block $\mathbf{c}$ and are independent from L, then using them we can compute sequentially the first p cofactors of the matrix $\tilde{L} : \tilde{\mathbf{L}}_1, \mathbf{D}_1, \tilde{\mathbf{L}}_2, \ldots, \mathbf{D}_{p-1}, \tilde{\mathbf{L}}_p$. From these we can write the first p cofactors of matrix L , can compute p columns of the matrix $\mathbf{d}L$ and after that the following p cofactors of matrix $\tilde{L}$, etc. For $p = 1$ we obtain a quasi-triangular matrix, that is a matrix with zero elements under the second diagonal, which is obtained by the elements $a_{2,1}, a_{3,2}, \ldots, a_{n,n-1}$.

Let us denote by A_k $(1 \leq k \leq n)$ the corner minors of order k of the quasi-triangular matrix $A = (a_{i,j}), a_{i,j} = 0$ for $i \geq 2, j \leq i-1$, and assume $\mathrm{A}_0 = 1$. Then its determinant can be computed as shown

$$\det(\mathrm{A}_n) = a_{nn} \det(\mathrm{A}_{n-1}) + \sum_{i=1}^{n-1} a_{i,n} \det(\mathrm{A}_{i-1}) \prod_{j=i+1}^{n-1} (-a_{j,j-1}).$$

The complexity of this method is $\frac{5}{3}n^3 + O(n^2)$ — multiplicative operations.

Example.

Let $A = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 0 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{pmatrix}$, then we obtain:

$$\tilde{\mathcal{L}}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{pmatrix}, \ \mathcal{L}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{(2)} = \tilde{\mathcal{L}}_1 A \mathcal{L}_1 = \begin{pmatrix} 0 & 7 & 0 & 1 & 1 \\ 2 & 6 & 1 & 0 & 2 \\ 0 & 12 & 2 & 0 & 0 \\ 0 & -4 & -1 & 2 & -2 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{L}}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{(3)} = \mathrm{diag}^{-1}(1,1,1,2,2)\tilde{\mathcal{L}}_2 A^{(2)} \mathcal{L}_2 = \begin{pmatrix} 0 & 7 & -4 & 1 & 1 \\ 2 & 6 & 12 & 0 & 2 \\ 0 & 12 & 24 & 0 & 0 \\ 0 & 0 & -72 & 12 & -12 \\ 0 & 0 & -144 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{L}}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 144 & -72 \end{pmatrix}, \quad \mathcal{L}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -72 & 0 \\ 0 & 0 & 0 & -144 & 1 \end{pmatrix},$$

$$A^{(4)} = \mathrm{diag}^{-1}(1,1,1,1,12)\tilde{\mathcal{L}}_2 A^{(2)} \mathcal{L}_2$$
$$= \begin{pmatrix} 0 & 7 & -4 & -216 & 1 \\ 2 & 6 & 12 & -288 & 2 \\ 0 & 12 & 24 & 0 & 0 \\ 0 & 0 & -72 & 864 & -12 \\ 0 & 0 & 0 & 10368 & -144 \end{pmatrix}.$$

The 1-triangular matrix $A^{(4)}$ is similar to $\mathrm{diag}(1,2,24,-864,-72)A$. The characteristic polynomial of the matrix A and the matrix $\mathrm{diag}(1,\frac{1}{2},\frac{1}{24},\frac{-1}{864},\frac{-1}{72})A^{(4)}$ is equal to $-x^5 + x^4 + 3x^3 - 22x^2 + 6x + 12$.

Часть IV

Conclusion

For computations over commutative domains we have the following results:

- The complexity of the $O(n^3)$ methods (FB) and (OP) for solving systems of linear equations of size $n \times m$ is

$$M_{(FB)} = (1/2)(4n^2m - 2n^3 - 2nm - 3n^2 - 2m + 7n - 2),$$

$$M_{(OP)} = (1/6)(12n^2m - 8n^3 - 6nm - 9n^2 - 12m + 8n + 12).$$

  Suppose that the complexity of the given method for matrix multiplications is $\gamma n^\beta + o(n^\beta)$, where γ and β are constants, and n is the order of the matrix. Then, the complexity of the recursive methods for solving systems of size $n \times m$ is

$$S(n, m) = \gamma \frac{n^\beta}{2^\beta}\left[(4\frac{m}{n} - 2)\frac{1 - n^{2-\beta}}{1 - 2^{2-\beta}} - \frac{1 - n^{1-\beta}}{1 - 2^{1-\beta}}\right] + o(n^{\beta-1}m).$$

- The complexity of the method for the computation of the determinant of a matrix of order n is $S(n, n)$. The complexity of the method for the computation of the kernel of a linear operator is $S(n, m)$.
- The complexity of the method for the computation and the factorization of the adjoint matrix is

$$F(n) = 6\gamma n^{\beta} \frac{1 - (n/2)^{1-\beta}}{2^{\beta} - 2} + o(n^{\beta})$$

- Finally, the complexity of the best method we know today for the computation of the characteristic polynomial of a matrix of order n is $\frac{5}{3}n^3 + O(n^2)$.

Часть V

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