Wavelets in NMR-imaging
Plan

- Wavelet theory:
  - From Fourier to wavelet analysis
  - Wavelet decomposition
  - Multiresolution analysis
- Applications
  - Signal denoising
  - Wavelet MRI encoding
Magnetic Resonance Imaging

- **Functional MRI**
  Used for detection of activation of brain regions as a reaction to different kinds of tasks.

- **MRI**
  Used for non-invasive observation of humans, animals, plants, insects and materials.
Problems in MRI

- Low signal to noise ratio in fMRI images
- Effects, rising due to long scanning time, e.g. subject movement
Wavelets

- From Fourier to wavelet analysis
- Wavelet decomposition
- Multiresolution analysis
From Fourier analysis

- **Fourier series**
  
  \[ f(x) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(kx) + b_k \sin(kx) \right) \]

- **Fourier coefficients**
  
  \[ a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) \, dx \]
  
  \[ a_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos(kx) \, dx \]
  
  \[ b_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin(kx) \, dx \]
Haar decomposition

- **Haar basis**
  \[ h(x) = \begin{cases} 
  1, & x \in [0, 1/2) \\
  -1, & x \in [1/2, 1) 
  \end{cases} \]

- **Haar series**
  \[
  h_0 = 1 \\
  h_{j,k} = 2^j h(2^j x - k) \\
  f(x) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \langle h_{j,k}(x), f(x) \rangle h_{j,k}
  \]
Wavelet analysis

- Wavelet basis
  \[ \int_0^\infty |\hat{\psi}(t\xi)| \frac{dt}{t} = 1 \]

- Shifts and scaling
  \[ \psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k) \]

- Coefficients and series
  \[ C_{j,k} = \langle f, \psi_{j,k} \rangle = 2^{\frac{j}{2}} \int f(x) \overline{\psi(2^j x - k)} \, dx \]
  \[ f(x) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{j,k} \psi_{j,k}(x) \]
Wavelet and Fourier representations
Multiresolution analysis

- Scaling and wavelet functions

- Series and coefficients

\[ f(x) = \sum_{k} A_{j_n,k} \varphi_{j_n,k} + \sum_{j \geq j_n,k} C_{j,k} \psi_{j,k} \]

\[ A_{j,k} = 2^{j/2} \int f(x) \varphi(2^j x - k) \, dx \]

\[ C_{j,k} = 2^{j/2} \int f(x) \psi(2^j x - k) \, dx \]
1D and 2D multiresolution analysis

- **1D**
  - Waveforms illustrating 1D multiresolution analysis.

- **2D**
  - Images illustrating 2D multiresolution analysis, showing different frequency bands (HL, LH, HH) at different scales.
Image denoising

- Simple thresholding
- Using probability of important detail presence
Noise suppression

- Threshold parameter
  \[ \lambda = \frac{1}{n_{\text{pix}}} \sum \Sigma^2(n) \]

- Soft and hard thresholding
  \[ C_{j,k} = \begin{cases} 
  0, & |C_{j,k}| < \lambda \\
  C_{j,k}, & |C_{j,k}| > \lambda 
\end{cases} \]
  \[ C_{j,k} = \begin{cases} 
  C_{j,k} - \lambda, & C_{j,k} > \lambda \\
  0, & |C_{j,k}| \leq \lambda \\
  C_{j,k} + \lambda, & C_{j,k} < \lambda 
\end{cases} \]
Examples

- Hard thresholding of fMRI data
Noise suppression, using probability of detail presence

- Wavelet decomposition
- Creation of the detail mask (by comparing original image and a supposed noise-free image)
- Computing the probability of detail presence along the mask
- Shrinking the coefficients regarding the probabilities
- Wavelet reconstruction
## Examples

<table>
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<tr>
<th>Source</th>
<th>Wavelet shrinkage</th>
<th>Gaussian smoothing</th>
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</thead>
<tbody>
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Wavelet MRI encoding

- Common MRI methods
- Wavelet encoding
Encoding of spin distribution.

- Frequency

- Phase
Idea of Fourier method

- Fourier encoding consists of acquiring FID on one direction (frequency encoding) and a FID analogue (phase encoding) on another.
- After that Fourier transform in each direction is applied to get image from the source data.
Fourier spatial encoding

- Phase (y) + frequency (x)
Idea of wavelet method

- As in Fourier encoding it is wise to acquire a 1D distribution through FID and Fourier transform (frequency encoding).
- The other dimension is scalar multiplied (in frequency domain) by wavelet basic functions, thus acquiring wavelet coefficients.
- Fourier and wavelet transforms are applied to the data in different directions to get an image.
Wavelet encoding basics

- Basic functions spectra – pulse envelope
Wavelet spatial encoding

- Wavelet (y) + frequency (x)
Experiment sequence

Fourier transform + Wavelet transform → Image
Examples

Fourier encoding
wavelet encoding
wavelet encoding (only 1/3 of original data)