# Analysis of an Algorithm Using the Hoare Logic

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Conversion into the hierarchical basis

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Outline of the presentation

### How can we approximate functions?

We want to approximate functions  $f : [a, b] \to \mathbb{R}$ . Simplification:  $f : [0, 1] \to \mathbb{R}, f(0) = f(1) = 0$ 



Problem ••••••••• Conversion into the nodal point basis  ${\tt 000000000}$ 

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Outline of the presentation

### How can we approximate functions?

We want to approximate functions  $f : [a, b] \to \mathbb{R}$ . Simplification:  $f : [0, 1] \to \mathbb{R}, f(0) = f(1) = 0$ 



"linear splines" with equidistant nodes ("lattice points") with distance  $h_n = 2^{-n}, n \in \mathbb{N}$  ("mesh size")

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Outline of the presentation

# Outline

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  - Algorithm for conversion into the nodal point basis
  - Proof of the algorithm
- 3 Conversion into the hierarchical basis
  - Algorithm for conversion into the hierarchical basis
  - Proof of the algorithm

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 Bases for the space of linear splines
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### The nodal point basis

We want to find a basis for the space of linear splines  $s : [0,1] \to \mathbb{R}$  with s(0) = s(1) = 0 and mesh size  $h_n = 2^{-n}$ . The lattice points are:

$$x_{n,i} = ih_n$$
 with  $i \in \{1, 2, ..., 2^n - 1\}$ 

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Bases for the space of linear splines

## The nodal point basis

We want to find a basis for the space of linear splines  $s : [0,1] \to \mathbb{R}$  with s(0) = s(1) = 0 and mesh size  $h_n = 2^{-n}$ . The lattice points are:

$$x_{n,i} = ih_n$$
 with  $i \in \{1, 2, ..., 2^n - 1\}$ 

A simple basis is:

$$\bigcup_{i=1}^{2^{n}-1} \{\Phi_{n,i}\} \text{ with } \Phi_{n,i} := \Phi(\frac{x - x_{n,i}}{h_{n}}), \Phi(x) := \max\{1 - |x|, 0\}$$



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Bases for the space of linear splines

## Representation in the nodal point basis

The  $\Phi_{n,i}$  are piecewise linear and continuous.  $\Phi_{n,i}(x_j) = \delta_{ij}$ . Piecewise linear and continuous functions are equal when they are equal on every lattice point

Conversion into the nodal point basis  ${\tt 000000000}$ 

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Bases for the space of linear splines

### Representation in the nodal point basis

The  $\Phi_{n,i}$  are piecewise linear and continuous.  $\Phi_{n,i}(x_j) = \delta_{ij}$ . Piecewise linear and continuous functions are equal when they are equal on every lattice point, so *s* can be expressed as follows:

$$s(x) = \sum_{i=1}^{2^{n}-1} f(x_i) \Phi_{n,i}(x)$$





Let's assume we would use the linear spline for quadrature (=numerical integration).

If we increase n, we have to compute everything again with the nodal point basis:



We will see, that there exists a better basis for this purpose.

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Bases for the space of linear splines

### Generating system which includes the nodal point basis

Instead of a basis for the linear splines we could also use a more general generating system:

$$\bigcup_{l=1}^{n}\bigcup_{i=1}^{2^l-1}\{\Phi_{l,i}\}$$

Conversion into the nodal point basis

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Bases for the space of linear splines

### Generating system which includes the nodal point basis

Instead of a basis for the linear splines we could also use a more general generating system:

$$\bigcup_{l=1}^{n}\bigcup_{i=1}^{2^{l}-1}\left\{ \Phi_{l,i}\right\}$$

A linear spline is represented in this generating system as follows:

$$s(x) = \sum_{l=1}^{n} \sum_{i=1}^{2^l-1} v_{l,i} \Phi_{l,i}(x)$$
 with a coefficient vector  $v$ 



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# The hierarchical basis

Generating system: 
$$\bigcup_{l=1}^{n}\bigcup_{i=1}^{2^{l}-1} \{\Phi_{l,i}\}$$

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We consider the following both bases:

• Nodal point basis: Only  $\Phi_{n,i}$  ( $\bigcup_{i=1}^{2^n-1} \{\Phi_{n,i}\}$ )

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# The hierarchical basis

Generating system: 
$$\bigcup_{l=1}^{n}\bigcup_{i=1}^{2^{l}-1} \{\Phi_{l,i}\}$$

We consider the following both bases:

- Nodal point basis: Only  $\Phi_{n,i}$   $(\bigcup_{i=1}^{2^n-1} \{\Phi_{n,i}\})$
- Hierarchical basis: Only  $\Phi_{I,i}$  with odd i $(\bigcup_{l=1}^{n} \bigcup_{i \in \{1,3,5,\dots,2^{l}-1\}} \{\Phi_{I,i}\})$



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Bases for the space of linear splines

# Quadrature with the hierarchical basis

Using the hierarchical basis, we don't need to recalculate the complete integral if we increase n:



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## Quadrature with the hierarchical basis

Using the hierarchical basis, we don't need to recalculate the complete integral if we increase n:



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Bases for the space of linear splines

# Outline

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Motivation		
Motivation		

In dimension 1, this problem is quite harmless, but it is a prototype for very complicated problems in higher dimensions. Keywords:

Sparse GridsFinite Elements

skip to revision of the Hoare rules



Figure: sparse grid

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Conversion into the nodal point basis

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Algorithm for conversion into the nodal point basis

# Representation in the nodal point basis

How can we represent the function given by the vector v in the nodal point basis? Algorithm toNodalPointBasis:

- input
  - integer n > 1
  - vector v with  $\sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i} = u$

output

• vector v with  $\sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i} = u$  and  $v_{l,i} = 0$  for all l < n

Conversion into the nodal point basis

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Algorithm for conversion into the nodal point basis

## Conversion into the nodal point basis

Algorithm for conversion of a vector v into the nodal point basis:

#### to Nodal Point Basis

for 
$$l = 1, ..., n - 1$$
:  
for  $i = 1, ..., 2^{l} - 1$ :  
 $v_{l+1,2i-1} += v_{l,i}/2$   
 $v_{l+1,2i} += v_{l,i}$   
 $v_{l+1,2i+1} += v_{l,i}/2$   
 $v_{l,i} = 0$ 

In the following we prove the correctness of the algorithm.

Conversion into the hierarchical basis

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Proof of the algorithm

### Revision of the Hoare rules

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Conversion into the hierarchical basis

Proof of the algorithm

## Revision of the Hoare rules

- $\bigcirc \{A\} \ \underline{\%} \text{ noOperation } \{A\}$
- **1** Axiom of assignment:  $\{A[E/x]\}x := E \{A\}$

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Proof of the algorithm

## Revision of the Hoare rules

- **O**  $\{A\}$  <u>%</u> noOperation  $\{A\}$
- **1** Axiom of assignment:  $\{A[E/x]\}x := E\{A\}$
- 2 Rule of consequence:  $\frac{A' \Rightarrow A \text{ and } \{A\} S\{B\} \text{ and } B \Rightarrow B'}{\{A'\} S\{B'\}}$

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Proof of the algorithm

## Revision of the Hoare rules

- **O**  $\{A\}$  <u>%</u> noOperation  $\{A\}$
- **1** Axiom of assignment:  $\{A[E/x]\}x := E\{A\}$
- 2 Rule of consequence:  $\frac{A' \Rightarrow A \text{ and } \{A\}S\{B\} \text{ and } B \Rightarrow B'}{\{A'\}S\{B'\}}$
- **3** Rule of composition:  $\frac{\{A\}S\{B\} \text{ and } \{B\}T\{C\}}{\{A\}S;T\{C\}}$

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Proof of the algorithm

## Revision of the Hoare rules

[A] % noOperation {A}
[Axiom of assignment: {A[E/x]}x := E {A}
[Aule of consequence: A' = A and {A}S{B} and B = B' {A'}S{B'}
[Aule of composition: {A}S{B} and {B}T{C} {A}S;T{C}
[Aule of iteration: {A and b}S{A} {A}while b do S{A and not(b)}

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Proof of the algorithm

### Revision of the Hoare rules

**0** {A}  $\frac{\%}{2}$  noOperation {A} **1** Axiom of assignment:  $\{A[E/x]\}x := E \{A\}$  **2** Rule of consequence:  $\frac{A' \Rightarrow A \text{ and } \{A\}S\{B\} \text{ and } B \Rightarrow B'}{\{A'\}S\{B'\}}$  **3** Rule of composition:  $\frac{\{A\}S\{B\} \text{ and } \{B\}T\{C\}}{\{A\}S;T\{C\}}$  **4** Rule of iteration:  $\frac{\{A \text{ and } b\}S\{A\}}{\{A\}\text{ while } b \text{ do } S\{A \text{ and } not(b)\}}$  **5** Conditional rule:  $\frac{\{A \text{ and } c\}S\{B\} \text{ and } \{A \text{ and } not(c)\}T\{B\}}{\{A\}\text{ if } c \text{ then } S \text{ else } T\{B\}}$ 

Problem	Conversion into the nodal point basis	Conversion into the hierarchical basis
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Proof of the algorithm		
Definitions		

We need the following definitions for the proof:

• 
$$f_{v} := \sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i}$$

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• 
$$P_u(f) :\Leftrightarrow f \equiv u$$
 (*u* is the input function)

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We need the following definitions for the proof:

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$$f_{v} := \sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i}$$

• 
$$P_u(f) :\Leftrightarrow f \equiv u$$
 (*u* is the input function)

• 
$$v_{l,i} := 0$$
 for  $l > n$  or  $l < 1$  or  $i < 1$  or  $i \ge 2^l$ 

Skip to the conversion into the hierarchical basis

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Proof of the algorithm

## Transformation of the program

The algorithm needs to be transformed so that it uses only control structures covered by the Hoare logic:

#### toNodalPointBasis

```
l = 1
while l \neq n:
    i = 1
    while i \neq 2^n:
        v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2
        v_{l+1,2i} = v_{l+1,2i} + v_{l,i}
        v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2
        v_{Li} = 0
        i = i + 1
    l = l + 1
```

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Proof of the algorithm

## Proof: The outer loop

 $\{P_{\mu}(f_{\nu})\}$ l = 1while  $l \neq n$ : i = 1while  $i \neq 2^{l}$ :  $v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$  $v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$  $v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$  $v_{l,i} = 0$ i = i + 1l = l + 1

 $\{P_u(f_v) \text{ and } \forall l' < n : v_{l',i'} = 0\}$  (proposition)

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Proof of the algorithm

## Proof: The outer loop

$$\{P_{u}(f_{v})\}\$$

$$l = 1$$

$$\{P_{u}(f_{v}) \text{ and } \forall l' < l : v_{l',i'} = 0\}$$
while  $l \neq n$ :
$$i = 1$$
while  $i \neq 2^{l}$ :
$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$

$$v_{l,i} = 0$$

$$i = i + 1$$

$$l = l + 1$$

 $\{P_u(f_v) \text{ and } \forall l' < n : v_{l',i'} = 0\}$  (proposition)

Conversion into the hierarchical basis

Proof of the algorithm

# Proof: The outer loop

$$\begin{cases} P_{u}(f_{v}) \\ l = 1 \\ \{P_{u}(f_{v}) \text{ and } \forall l' < l : v_{l',i'} = 0 \} \\ while \ l \neq n : \\ \{P_{u}(f_{v}) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \} \\ i = 1 \\ while \ i \neq 2^{l} : \\ v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2 \\ v_{l+1,2i} = v_{l+1,2i} + v_{l,i} \\ v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2 \\ v_{l,i} = 0 \\ i = i + 1 \\ l = l + 1 \\ \{P_{u}(f_{v}) \text{ and } \forall l' < l : v_{l',i'} = 0 \} \\ \{P_{u}(f_{v}) \text{ and } \forall l' < n : v_{l',i'} = 0 \} \text{ (follows)}$$

Conversion into the nodal point basis

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Proof of the algorithm

# Proof: The inner loop

$$\{P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n\}$$
  
 $i = 1$ 

while  $i \neq 2^{l}$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$l = l + 1$$
  
 $\{P_u(f_v) \text{ and } orall l' < l : v_{l',i'} = 0\}$  (proposition)

Conversion into the hierarchical basis

Proof of the algorithm

# Proof: The inner loop

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$
  

$$i = 1$$
  

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n \}$$
  
while  $i \neq 2^l$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$l = l + 1$$
  
 $\{P_u(f_v) \text{ and } orall l' < l : v_{l',i'} = 0\}$  (proposition)

Conversion into the hierarchical basis

Proof of the algorithm

# Proof: The inner loop

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$
  

$$i = 1$$
  

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n \}$$
  
while  $i \neq 2^l$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' \leq l : v_{l',i'} = 0 \}$$
 (prop.)  
 
$$l = l + 1$$
  
 
$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \}$$
 (follows)

Conversion into the hierarchical basis

Proof of the algorithm

# Proof: The inner loop

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$
  

$$i = 1$$
  

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n \}$$
  
while  $i \neq 2^l$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < 2^l : v_{l,i'} = 0 \} \text{ (prop.)}$$
  
 
$$I = l + 1$$
  
 
$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \} \text{ (follows)}$$
Conversion into the hierarchical basis

Proof of the algorithm

$$X := orall I' < I : v_{I',i'} = 0$$
 and  $orall i' < i : v_{I,i'} = 0$  and  $I \neq n$ 

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$
  

$$i = 1$$
  

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n \}$$
  

$$while i \neq 2^l :$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < 2^l : v_{l,i'} = 0 \} \text{ (prop.)}$$
  
 
$$I = l + 1$$
  
 
$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \} \text{ (follows)}$$

Conversion into the hierarchical basis

Proof of the algorithm

$$X := \forall l' < l : v_{l',i'} = 0$$
 and  $\forall i' < i : v_{l,i'} = 0$  and  $l \neq n$ 

$$\{P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$
  

$$i = 1$$
  

$$\{P_u(f_v) \text{ and } X \}$$
  
while  $i \neq 2^l$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$
  

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$
  

$$v_{l,i} = 0$$
  

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < 2^l : v_{l,i'} = 0 \} \text{ (prop.)}$$
  
 
$$I = l + 1$$
  
 
$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \} \text{ (follows)}$$

Conversion into the hierarchical basis

Proof of the algorithm

$$X := orall I' < I : v_{I',i'} = 0$$
 and  $orall i' < i : v_{I,i'} = 0$  and  $I \neq n$ 

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } l \neq n \}$$

$$i = 1$$

$$\{ P_u(f_v) \text{ and } X \}$$
while  $i \neq 2^l$ :
$$\{ P_u(f_v) \text{ and } X \text{ and } i \neq 2^l \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$

$$v_{l,i} = 0$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < 2^l : v_{l,i'} = 0 \}$$

$$(follows)$$

$$l = l + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' < l : v_{l',i'} = 0 \}$$

$$(follows)$$

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Proof of the algorithm

#### Proof: The inner block

$$X := \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n$$

$$\{P_u(f_v) \text{ and } X\}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$

 $v_{I,i} = 0$ 

i = i + 1{ $P_u(f_v)$  and X} (proposition)

Conversion into the nodal point basis 0000000000

Conversion into the hierarchical basis

Proof of the algorithm

#### Proof: The inner block

$$X := \forall l' < l : v_{l',i'} = 0 \text{ and } \forall i' < i : v_{l,i'} = 0 \text{ and } l \neq n$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$

$$\begin{aligned} &v_{I,i} = 0 \\ &\{P_u(f_v) \text{ and } X \text{ and } v_{I,i} = 0\} \text{ (proposition)} \\ &i = i + 1 \\ &\{P_u(f_v) \text{ and } X\} \text{ (follows)} \end{aligned}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

#### Proof: The inner block

$$X := \forall l' < l : v_{l',i'} = 0$$
 and  $\forall i' < i : v_{l,i'} = 0$  and  $l \neq n$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$

$$\begin{aligned} &v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2 \\ &\{P_u(f_v - v_{l,i}\Phi_{l,i}) \text{ and } X\} \text{ (proposition)} \\ &v_{l,i} = 0 \\ &\{P_u(f_v) \text{ and } X \text{ and } v_{l,i} = 0\} \text{ (follows)} \\ &i = i+1 \\ &\{P_u(f_v) \text{ and } X\} \text{ (follows)} \end{aligned}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

#### Proof: The inner block

$$X := \forall l' < l : v_{l',i'} = 0$$
 and  $\forall i' < i : v_{l,i'} = 0$  and  $l \neq n$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$\begin{array}{l} v_{l+1,2i} = v_{l+1,2i} + v_{l,i} \\ \left\{ P_u(f_v + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2)) \text{ and } X \right\} \text{ (proposition)} \\ v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2 \\ \left\{ P_u(f_v - v_{l,i}\Phi_{l,i}) \text{ and } X \right\} \text{ (follows)} \\ v_{l,i} = 0 \\ \left\{ P_u(f_v) \text{ and } X \text{ and } v_{l,i} = 0 \right\} \text{ (follows)} \\ i = i + 1 \\ \left\{ P_u(f_v) \text{ and } X \right\} \text{ (follows)} \end{array}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Proof of the algorithm

#### Proof: The inner block

$$X := orall I' < I : v_{I',i'} = 0$$
 and  $orall i' < i : v_{I,i'} = 0$  and  $I \neq n$ 

$$\begin{split} &v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2 \\ & \{P_u(f_v + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2 + \Phi_{l+1,2i})) \text{ and } X\} \text{ (proposition)} \\ &v_{l+1,2i} = v_{l+1,2i} + v_{l,i} \\ & \{P_u(f_v + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2)) \text{ and } X\} \text{ (follows)} \\ &v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2 \\ & \{P_u(f_v - v_{l,i}\Phi_{l,i}) \text{ and } X\} \text{ (follows)} \\ &v_{l,i} = 0 \\ & \{P_u(f_v) \text{ and } X \text{ and } v_{l,i} = 0\} \text{ (follows)} \\ &i = i + 1 \\ & \{P_u(f_v) \text{ and } X\} \text{ (follows)} \end{split}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Proof of the algorithm

$$X := orall I' < I : v_{I',i'} = 0$$
 and  $orall i' < i : v_{I,i'} = 0$  and  $I \neq n$ 

$$\{P_{u}(f_{v}) \text{ and } X\}$$

$$\{P_{u}(f_{v} + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2 + \Phi_{l+1,2i} + \Phi_{l+1,2i-1}/2)) \text{ and } X\}$$
(proposition)
$$v_{l+1,2i-1} = v_{l+1,2i-1} + v_{l,i}/2$$

$$\{P_{u}(f_{v} + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2 + \Phi_{l+1,2i})) \text{ and } X\}$$
(follows)
$$v_{l+1,2i} = v_{l+1,2i} + v_{l,i}$$

$$\{P_{u}(f_{v} + v_{l,i}(-\Phi_{l,i} + \Phi_{l+1,2i+1}/2)) \text{ and } X\}$$
(follows)
$$v_{l+1,2i+1} = v_{l+1,2i+1} + v_{l,i}/2$$

$$\{P_{u}(f_{v} - v_{l,i}\Phi_{l,i}) \text{ and } X\}$$
(follows)
$$v_{l,i} = 0$$

$$\{P_{u}(f_{v}) \text{ and } X \text{ and } v_{l,i} = 0\}$$
(follows)
$$i = i + 1$$

$$\{P_{u}(f_{v}) \text{ and } X\}$$
(follows)

Problem 000000000	Conversion into the nodal point basis	Conversion into the hierarchical basis
Proof of the algorithm		
Proof: End		

The following assertions are equivalent:

• 
$$\{P_u(f_v) \text{ and } X\}$$

$$= \{ P_u(f_v + V_{l,i}(\underbrace{-\Phi_{l,i} + \Phi_{l+1,2i+1}/2 + \Phi_{l+1,2i} + \Phi_{l+1,2i-1}/2}_{=0}) \text{ and } X \}$$

Problem 000000000	Conversion into the nodal point basis	Conversion into the hierarchical basis
Proof of the algorithm		
Proof: End		

The following assertions are equivalent:

• 
$$\{P_u(f_v) \text{ and } X\}$$

$$= \{P_u(f_v + V_{l,i}(\underbrace{-\Phi_{l,i} + \Phi_{l+1,2i+1}/2 + \Phi_{l+1,2i} + \Phi_{l+1,2i-1}/2}_{=0})) \text{ and } X\}$$

The algorithm terminates in all cases because it consists only of for loops.

qed

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Algorithm for conversion into the hierarchical basis

### Representation in the hierarchical basis

How can we represent the function given by the vector v in the hierarchical basis?

Algorithm toHierarchicalBasis:

- input
  - integer n > 1
  - vector v with  $\sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i} = u$

output

• vector 
$$v$$
 with  

$$\sum_{l=1}^{n} \sum_{i=1}^{2^{l}-1} v_{l,i} \Phi_{l,i} = u \text{ and } v_{l,i} = 0 \text{ for all even } i$$

Conversion into the nodal point basis

Conversion into the hierarchical basis 0

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Algorithm for conversion into the hierarchical basis

#### Conversion into the hierarchical basis

Algorithm for conversion of a vector v into the hierarchical basis:

toHierarchicalBasis (wrong!)

for 
$$l = n - 1, ..., 1$$
:  
for  $i = 1, ..., 2^{l} - 1$ :  
 $v_{l+1,2i-1} = v_{l+1,2i}/2$   
 $v_{l+1,2i+1} = v_{l+1,2i}/2$   
 $v_{l,i} = v_{l+1,2i}$   
 $v_{l+1,2i} = 0$ 

In the following we prove the correctness of the (corrected) algorithm.

Conversion into the nodal point basis

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Algorithm for conversion into the hierarchical basis

## Transformation of the program

We have to transform our program so that it uses only the control structures covered by the Hoare logic:

toHierarchicalBasis (wrong!)
l = n - 1
while $I \neq 0$ :
i = 1
while $i \neq 2^{l}$ :
$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$
$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$
$v_{l,i} = v_{l+1,2i}$
$v_{l+1,2i}=0$
i = i + 1
l = l - 1

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

#### Proof: The outer loop

 $\{P_{\mu}(f_{\nu})\}$ l = n - 1while  $I \neq 0$ : i = 1while  $i \neq 2^{l}$ :  $v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$  $v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$  $v_{l,i} = v_{l+1,2i}$  $v_{l+1,2i} = 0$ i = i + 1l = l - 1

 $\{P_u(f_v) \text{ and } \forall l' > 1 : v_{l',2i'=0}\}$  (proposition)

Conversion into the nodal point basis

Proof of the algorithm

### Proof: The outer loop

$$\{P_{u}(f_{v})\}\$$

$$l = n - 1$$

$$\{P_{u}(f_{v}) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0\}$$
while  $l \neq 0$ :
$$i = 1$$
while  $i \neq 2^{l}$ :
$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$v_{l,i} = v_{l+1,2i}$$

$$v_{l+1,2i} = 0$$

$$i = i + 1$$

$$l = l - 1$$

 $\{P_u(f_v) \text{ and } \forall l' > 1 : v_{l',2i'=0}\}$  (proposition)

Conversion into the nodal point basis

Conversion into the hierarchical basis  $\circ\circ\circ\bullet\circ\circ\circ\circ\circ\circ$ 

Proof of the algorithm

# Proof: The outer loop

$$\begin{cases} P_{u}(f_{v}) \\ l = n - 1 \\ \{P_{u}(f_{v}) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0 \} \\ \text{while } l \neq 0 : \\ \{P_{u}(f_{v}) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0 \text{ and } l \neq 0 \} \\ i = 1 \\ \text{while } i \neq 2^{l} : \\ v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2 \\ v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2 \\ v_{l,i} = v_{l+1,2i} \\ v_{l+1,2i} = 0 \\ i = i + 1 \\ l = l - 1 \\ \{P_{u}(f_{v}) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0 \} \\ \{P_{u}(f_{v}) \text{ and } \forall l' > 1 : v_{l',2i'=0}\} \text{ (follows)}$$

Conversion into the nodal point basis  ${\scriptstyle 00000000}$ 

Proof of the algorithm

## Proof: The inner loop

$$\{P_u(f_v) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0 \text{ and } l \neq 0\}$$
  
 $i = 1$ 

while  $i \neq 2^{l}$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i/2}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i/2}$$
  

$$v_{l,i} = v_{l+1,2i}$$
  

$$v_{l+1,2i} = 0$$
  

$$i = i + 1$$

$$l = l - 1$$
  
 $\{P_u(f_v) \text{ and } orall l' > l + 1 : v_{l',2i'} = 0\}$  (proposition)

Conversion into the nodal point basis

Proof of the algorithm

## Proof: The inner loop

$$X := orall l' > l+1: v_{l',2i'} = 0$$
 and  $orall i' < i: v_{l+1,2i'} = 0$  and  $l 
eq 0$ 

$$\{P_u(f_v) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0 \text{ and } l \neq 0\}$$
  
 $i = 1$ 

while  $i \neq 2^{l}$ :

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i/2}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i/2}$$

$$v_{l,i} = v_{l+1,2i}$$

$$v_{l+1,2i} = 0$$

$$i = i + 1$$

$$l = l - 1$$
  
 $\{P_u(f_v) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0\}$  (proposition)

Conversion into the nodal point basis

Proof of the algorithm

$$X:= orall l' > l+1: v_{l',2i'}=0$$
 and  $orall i' < i: v_{l+1,2i'}=0$  and  $l 
eq 0$ 

$$\begin{aligned} &\{P_u(f_v) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0 \text{ and } l \neq 0 \\ &i = 1 \\ &\{P_u(f_v) \text{ and } X \} \\ & \text{while } i \neq 2^l : \end{aligned}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i/2}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i/2}$$
  

$$v_{l,i} = v_{l+1,2i}$$
  

$$v_{l+1,2i} = 0$$
  

$$i = i + 1$$

$$l = l - 1$$
  
 $\{P_u(f_v) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0\}$  (proposition)

Conversion into the nodal point basis

Proof of the algorithm

$$X:= orall l' > l+1: v_{l',2i'}=0$$
 and  $orall i' < i: v_{l+1,2i'}=0$  and  $l 
eq 0$ 

$$\begin{aligned} &\{P_u(f_v) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0 \text{ and } l \neq 0 \\ &i = 1 \\ &\{P_u(f_v) \text{ and } X \} \\ & \text{while } i \neq 2^l : \end{aligned}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i/2}$$
  

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i/2}$$
  

$$v_{l,i} = v_{l+1,2i}$$
  

$$v_{l+1,2i} = 0$$
  

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } \forall l' > l : v_{l',2i'} = 0 \} \text{ (proposition)}$$
  
 
$$l = l - 1$$
  
 
$$\{ P_u(f_v) \text{ and } \forall l' > l + 1 : v_{l',2i'} = 0 \} \text{ (follows)}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$X:= orall I' > l+1: v_{l',2i'}=0$$
 and  $orall i' < i: v_{l+1,2i'}=0$  and  $l 
eq 0$ 

```
\{P_{ij}(f_v) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0 \text{ and } l \neq 0\}
i = 1
\{P_{\mu}(f_{\nu}) \text{ and } X\}
while i \neq 2':
     \{P_{\mu}(f_{\nu}) \text{ and } X\}
     v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2
     v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2
                                                               proposition
     v_{l,i} = v_{l+1,2i}
     v_{l+1,2i} = 0
     i = i + 1
     \{P_{\mu}(f_{\nu}) \text{ and } X\}
\{P_{ii}(f_v) \text{ and } \forall l' > l : v_{l'.2i'} = 0\} (follows)
l = l - 1
\{P_{\mu}(f_{\nu}) \text{ and } \forall l' > l+1 : v_{l',2i'} = 0\} (follows)
```

Proof of the algorithm

## Proof: The inner block (1)

$$X := orall l' > l + 1 : v_{l',2i'} = 0$$
 and  $orall i' < i : v_{l+1,2i'} = 0$  and  $l \neq 0$ 

 $\{P_u(f_v) \text{ and } X\}$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

 $v_{l,i} = v_{l+1,2i}$ 

 $v_{l+1,2i} = 0$ 

i = i + 1{ $P_u(f_v)$  and X}

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Proof of the algorithm

## Proof: The inner block (1)

$$X := \forall l' > l+1 : v_{l',2i'} = 0$$
 and  $\forall i' < i : v_{l+1,2i'} = 0$  and  $l \neq 0$ 

 $\{P_u(f_v) \text{ and } X\}$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

 $v_{l,i} = v_{l+1,2i}$ 

$$egin{aligned} &v_{l+1,2i} = 0 \ &\{P_u(f_v) ext{ and } X ext{ and } v_{l+1,2i} = 0\} \ &i = i+1 \ &\{P_u(f_v) ext{ and } X\} \end{aligned}$$

Conversion into the nodal point basis

Proof of the algorithm

## Proof: The inner block (1)

$$X := \forall l' > l + 1 : v_{l',2i'} = 0$$
 and  $\forall i' < i : v_{l+1,2i'} = 0$  and  $l \neq 0$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\begin{aligned} v_{l,i} &= v_{l+1,2i} \\ \{ P_u(f_v - v_{l+1,2i} \Phi_{l+1,2i}) \text{ and } X \} \\ v_{l+1,2i} &= 0 \\ \{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \} \\ i &= i+1 \\ \{ P_u(f_v) \text{ and } X \} \end{aligned}$$

Conversion into the nodal point basis

Proof of the algorithm

### Proof: The inner block (1)

$$X := \forall l' > l + 1 : v_{l',2i'} = 0$$
 and  $\forall i' < i : v_{l+1,2i'} = 0$  and  $l \neq 0$ 

 $\{P_u(f_v) \text{ and } X\}$ 

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\begin{split} v_{l+1,2i+1} &= v_{l+1,2i+1} - v_{l+1,2i}/2 \\ \{ P_u(f_v + v_{l+1,2i}) + v_{l+1,2i} - v_{l,i} \Phi_{l,i} \} \text{ and } X \} \\ v_{l,i} &= v_{l+1,2i} \\ \{ P_u(f_v - v_{l+1,2i} \Phi_{l+1,2i}) \text{ and } X \} \\ v_{l+1,2i} &= 0 \\ \{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \} \\ i &= i+1 \\ \{ P_u(f_v) \text{ and } X \} \end{split}$$

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Conversion into the nodal point basis

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Proof of the algorithm

### Proof: The inner block (1)

$$X := orall l' > l+1: v_{l',2i'} = 0$$
 and  $orall i' < i: v_{l+1,2i'} = 0$  and  $l \neq 0$ 

$$\begin{aligned} & v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2 \\ & \{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \} \\ & v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2 \\ & \{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \} \\ & v_{l,i} = v_{l+1,2i} \\ & \{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } X \} \\ & v_{l+1,2i} = 0 \\ & \{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \} \\ & i = i + 1 \\ & \{ P_u(f_v) \text{ and } X \} \end{aligned}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

4

Proof of the algorithm

$$X:=orall I'>I+1:v_{I',2i'}=0$$
 and  $orall i'< i:v_{I+1,2i'}=0$  and  $I
eq 0$ 

$$\{P_u(f_v) \text{ and } X\}$$

$$\{P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X\}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X\}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X\}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } X\}$$

$$v_{l+1,2i} = 0$$

$$\{P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0\}$$

$$i = i + 1$$

$$\{P_u(f_v) \text{ and } X\}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Proof of the algorithm

# Proof: The inner block (2)

We would have to show that the following assertions are equivalent:

- $\{P_u(f_v) \text{ and } X\}$
- { $P_u(f_v + v_{l+1,2i}(\Phi_{l,i} \Phi_{l+1,2i} \Phi_{l+1,2i+1}/2 \Phi_{l+1,2i-1}/2) v_{l,i}\Phi_{l,i})$  and X}

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Proof of the algorithm

## Proof: The inner block (2)

We would have to show that the following assertions are equivalent:

- $\{P_u(f_v) \text{ and } X\}$
- $\{P_u(f_v + v_{l+1,2i}(\Phi_{l,i} \Phi_{l+1,2i} \Phi_{l+1,2i+1}/2 \Phi_{l+1,2i-1}/2) v_{l,i}\Phi_{l,i}\}$

We already know that

$$\Phi_{l,i} = \frac{\Phi_{l+1,2i-1}}{2} + \Phi_{l+1,2i} + \frac{\Phi_{l+1,2i+1}}{2}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

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Proof of the algorithm

## Proof: The inner block (2)

We would have to show that the following assertions are equivalent:

• 
$$\{P_u(f_v) \text{ and } X\}$$

 $\{P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}\}$ 

We already know that

$$\Phi_{l,i} = \frac{\Phi_{l+1,2i-1}}{2} + \Phi_{l+1,2i} + \frac{\Phi_{l+1,2i+1}}{2}$$

So something is wrong with  $-v_{I,i}\Phi_{I,i}$ .

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

Conversion into the nodal point basis

Conversion into the hierarchical basis

Proof of the algorithm

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i}) - v_{l,i}\Phi_{l,i}) \text{ and } X \}$$

$$v_{l,i} = v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$
Conversion into the nodal point basis  ${\tt 000000000}$ 

Conversion into the hierarchical basis

Proof of the algorithm

# Proof: The inner block (corrected)

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2)) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2)) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i})) \text{ and } X \}$$

$$v_{l,i} = v_{l,i} + v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

Proof of the algorithm

# Proof: The inner block (corrected)

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v) \text{ and } X \}$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2 - \Phi_{l+1,2i-1}/2)) \text{ and } X \}$$

$$v_{l+1,2i-1} = v_{l+1,2i-1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i} - \Phi_{l+1,2i+1}/2)) \text{ and } X \}$$

$$v_{l+1,2i+1} = v_{l+1,2i+1} - v_{l+1,2i}/2$$

$$\{ P_u(f_v + v_{l+1,2i}(\Phi_{l,i} - \Phi_{l+1,2i})) \text{ and } X \}$$

$$v_{l,i} = v_{l,i} + v_{l+1,2i}$$

$$\{ P_u(f_v - v_{l+1,2i}\Phi_{l+1,2i}) \text{ and } < X \}$$

$$v_{l+1,2i} = 0$$

$$\{ P_u(f_v) \text{ and } X \text{ and } v_{l+1,2i} = 0 \}$$

$$i = i + 1$$

$$\{ P_u(f_v) \text{ and } X \}$$

qed

Conversion into the nodal point basis  ${\tt 000000000}$ 

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Proof of the algorithm

## Coversion into the the hierarchical basis (corrected)

So we have the following corrected algorithm for converting a vector v into the hierarchical basis:

#### toHierarchicalBasis

for 
$$l = n - 1, ..., 1$$
:  
for  $i = 1, ..., 2^{l} - 1$ :  
 $v_{l+1,2i-1} = v_{l+1,2i}/2$   
 $v_{l+1,2i+1} = v_{l+1,2i}/2$   
 $v_{l,i} + v_{l+1,2i}$   
 $v_{l+1,2i} = 0$ 

Obviously the algorithm terminates in all cases :)

Conversion into the nodal point basis

Conversion into the hierarchical basis  $\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ$ 

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Proof of the algorithm

#### Conclusion

We have seen:

- How we can prove the correctness of an algorithm
- How we can find bugs with the Hoare logic

Conversion into the nodal point basis

Conversion into the hierarchical basis  $\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ$ 

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Proof of the algorithm

### Conclusion

We have seen:

- How we can prove the correctness of an algorithm
- How we can find bugs with the Hoare logic

This presentation is based on a presentation by Samuel Kerschbaumer.

Conversion into the nodal point basis

Conversion into the hierarchical basis  $\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ$ 

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Proof of the algorithm

## Conclusion

We have seen:

- How we can prove the correctness of an algorithm
- How we can find bugs with the Hoare logic

# Thank you for your attention!

This presentation is based on a presentation by Samuel Kerschbaumer.