Numerical Simulation of Pressure Surge with the Method of Characteristics

R. Fiereder

02.04.2009 Saint Petersburg, Russia
Content

- Motivation
- Governing Equations
  - Continuity Equation
  - Momentum Equation
- Method of Characteristics
- Implementation
- Boundary Conditions
- Application
  - pressure surge
  - oscillating valve
  - surge chamber oscillations
  - power plant simulation
Motivation

- Simulation of interaction of plant components
- Identification of resonance cases
Motivation
Governing Equations

- 1D
- unsteady
- compressible

- Continuity equation
- Momentum equation
Continuity Equation

\[
\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{A} \frac{dA}{dt} + \frac{\partial v}{\partial x} = 0
\]

- Continuity
  \[
  \frac{\partial (\rho A)}{\partial t} + \frac{\partial (\rho v A)}{\partial x} = 0
  \]
- Product rule
  \[
  A \frac{\partial \rho}{\partial t} + \rho \frac{\partial A}{\partial t} + A v \frac{\partial \rho}{\partial x} + \rho v \frac{\partial A}{\partial x} + \rho A \frac{\partial v}{\partial x} = 0
  \]
- Total differential
  \[
  \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x}
  \]
Continuity Equation

\[ \frac{1}{A} \frac{dA}{dt} = \frac{2R}{E_R s} \left(1 - \mu^2\right) \frac{dp}{dt} \]

- Circular Pipe
  \[ \frac{dA}{dt} = 2\pi R \frac{dR}{dt} \quad \frac{1}{A} \frac{dA}{dt} = 2 \frac{d\varepsilon_\phi}{dt} \]
- No axial displacement / Hooke’s law
  \[ \sigma_a = \mu \sigma_\phi \quad \varepsilon_\phi = \frac{\sigma_\phi - \mu \sigma_a}{E_R} \]
- \( s << R \)
  \[ \frac{d\sigma_\phi}{dt} \approx \frac{R}{s} \frac{dp}{dt} \]
Continuity Equation

\[ \frac{1}{\rho} \frac{d\rho}{dt} = \frac{1}{E_F} \frac{dp}{dt} \]

- Fluid compressibility
  - Hooke's law applies
    \[ E_F = \frac{dp}{dV/V} \]
  - Mass in Control Volume is constant
    \[ dm = Vd\rho + \rho dV = 0 \]
    \[ \frac{dV}{V} = -\frac{d\rho}{\rho} \]
Continuity Equation

\[ \frac{\partial p}{\partial t} + \nu \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} = 0 \]

\[ \frac{dp}{dt} \left[ \frac{1}{E_F} + \frac{2R}{E_R} \left(1 - \mu^2\right) \right] + \frac{\partial v}{\partial x} = 0 \]

\[ a^2 = \frac{E_{sys}}{\rho} \]
Momentum Equation

\[
\frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{\partial z}{\partial x} + gJ_e + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0
\]

- Momentum conservation
  \[\sum_i F_i = \frac{dI}{dt}\]

- Pressure force
  \[F_p = -A \frac{\partial p}{\partial x} dx\]

- Gravity force
  \[F_g = -\rho g A \frac{\partial z}{\partial x} dx\]

- Friction force
  \[F_\tau = -2\pi R \tau_0 dx\]

\[\tau_0 = -\frac{\rho \lambda |v|^4}{8}\]

\[\lambda = -\frac{4J_e gR}{v^2}\]
Final Set of Equations

\[
\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} = 0
\]
\[
\frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{\partial z}{\partial x} + g J_e + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0
\]

- First order hyperbolic differential equation system
- Two dependent variables p, v
- Two independent variables x, t
- a, \rho, J_e are parameters of the system
- Solution by means of Method of Characteristics
Method of Characteristics

- Hyperbolic differential equation system
  - Interferences propagate at a certain speed
  - Existence of constant variables along characteristic lines:

\[
\frac{d}{dt} R^\pm \bigg|_{c^*} = 0
\]

Riemann Variables

\[
\begin{aligned}
\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial v}{\partial x} &= G_1 = 0 \\
\frac{1}{\rho} \frac{\partial p}{\partial x} + g \frac{\partial z}{\partial x} + g J_e + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} &= G_2 = 0 \\
G &= G_2 + \xi G_1
\end{aligned}
\]

\[
\begin{aligned}
\left[ \frac{\partial v}{\partial t} + (v + \xi \rho a^2) \frac{\partial v}{\partial x} \right] + \xi \left[ \frac{\partial p}{\partial t} + \left(v + \frac{1}{\rho \xi}\right) \frac{\partial p}{\partial x} \right] &= 0 \\
\frac{dv}{dt} + \xi \frac{dp}{dt} + g \left(J_e + \frac{\partial z}{\partial x}\right) &= 0 \\
dx/dt &= v + \xi \rho a^2 = \frac{1}{\rho \xi} \\
\xi_{*} &= \pm \frac{1}{\rho a}
\end{aligned}
\]
Method of Characteristics

\[
\frac{dv}{dt} + \frac{1}{\rho a} \frac{dp}{dt} + g \left( J_e + \frac{\partial z}{\partial x} \right) = \frac{d}{dt} R^+ = 0
\]

\[
C^+ = \frac{dx}{dt} = v + a
\]

\[
\frac{dv}{dt} - \frac{1}{\rho a} \frac{dp}{dt} + g \left( J_e + \frac{\partial z}{\partial x} \right) = \frac{d}{dt} R^- = 0
\]

\[
C^- = \frac{dx}{dt} = v - a
\]
Method of Characteristics

- Calculate values in P from known values in Point A and B
  
  - Integrate $R^\pm$ along $C^\pm$
    
    $\int_A^P dv + \frac{1}{\rho A} \int_A^P dp + g \int_A^P J_e dt + g \int_A^P \frac{\partial Z}{\partial x} dt = 0$
    
    $\int_B^P dv - \frac{1}{\rho A} \int_B^P dp + g \int_B^P J_e dt + g \int_B^P \frac{\partial Z}{\partial x} dt = 0$
  
  - Two equations for two unknown variables
Method of Characteristics - Summary

- Solution is only valid on Characteristics
- Area of influence
- Area of dependence
- Characteristics separate two different solution domains
Implementation

- Simplifications
  - $a >> v$
  - $C^- = \frac{dx}{dt} = a \quad \rightarrow \quad \Delta x = a \Delta t$
  - velocity $v$ is not known along $C$
  - Friction term has to be approximated

$$\int_P J_v \, dt = J \int v^2 \, dt \cong J v_A |v_A| \Delta t + O(\Delta t) = J v_A \Delta t$$

- Required
  - Computational grid
  - Steady-state solution for the system
  - Boundary condition providing a disturbance
Implementation

- Get system parameters
- Initialize system
  - Calculate Wave velocity $a$
  - Generate calculation grid
- Calculate steady state solution
  - Stationary Hydraulics
- Calculate solution for time $t$

```
read in parameters
initialize system
calculate steady state solution
t = t_0
calculate solution at time t
write results? Yes
  write results to file
No
\[ t += \Delta t \]
\[ t < t_{\text{max}}? \]
Yes
No
Stop
```
Implementation

- Method for calculating new time level
  - $C^+$
    \[
    (v_p - v_A) + \frac{1}{\rho a} (p_p - p_A) + gJ_{e,A}\Delta t = 0
    \]
  - $C^-$
    analog
    \[
    v_p + \frac{1}{\rho a} p_p - v_A - \frac{1}{\rho a} p_A + gJ_{e,A}\Delta t = 0
    \]
    \[
    K_A^{+} + \frac{K_B^{-}}{2}
    \]
    \[
    v_p = \frac{K_B^{-} - K_A^{+}}{2}
    \]
Implementation

- Calculate $\lambda$
- Evaluate boundary condition
- Calculate solution for inner field
- Calculate $K^+$ and $K^-$

Flowchart:

1. Call from main
2. Calculate $\lambda$ for friction term
3. Boundary condition left end $i = 0$
4. Boundary condition right end $i = i_{\text{max}}$
5. $i = 1$
6. Calculate $v$ and $p$ for point $i$
7. $i++$
8. $i < i_{\text{max}}$?
   - Yes: continue
   - No: $i = 0$
9. Calculate $K^+$ and $K^-$ for point $i$
10. $i > i_{\text{max}}$?
    - Yes: back to main
    - No: $i++$
Boundary Conditions

- **Static**
  - passive → Reservoir
  - active → Valve

- **Dynamic**
  - passive → surge chamber
  - active → Francis Turbine

<table>
<thead>
<tr>
<th>Passive</th>
<th>Active</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static</td>
<td>( f(p, v) )</td>
</tr>
<tr>
<td>Dynamic</td>
<td>( f(p, \frac{dp}{dt}, \frac{dv}{dt}) )</td>
</tr>
</tbody>
</table>
Boundary Conditions

- **Reservoir**
  - \( p = \text{const.} \rightarrow v \text{ is calculated} \)
  - compression waves are reflected as expansions waves
  - expansions waves are reflected as compression waves

- **Wall**
  - \( v = 0 \rightarrow p \text{ is calculated} \)
  - waves are reflected
Boundary Conditions

- **Valve**
  
  \[ v = f(\Delta p, \tau, \zeta) \]

  - \( \zeta \) -> loss coefficient

  \[ \tau = \frac{A_{\text{valve}}}{A_{\text{pipe}}} \]

  \[ v_1 = \zeta \tau \sqrt{\frac{2}{\rho} \cdot |p_0 - p_1|} \]
Boundary Conditions

- Surge chamber
  - $p_W$ cannot be computed directly
    - $Q_W(h_w,\ldots)$
  - Predictor corrector scheme
    - Guess for $p_W$
    - Calculate $Q_W$
    - Correct $p_W$
    - Correct $Q_W$

\[
\Delta h_W = h_{W,t_0 + \Delta t} - h_{W,t_0}
\]

\[
\frac{dh}{dt} = \frac{1}{A_w(h)} (Q_{W,L} - Q_{W,R})
\]
Boundary Conditions

- Turbine
  - Machine data
    - Mass moment of inertia of machine and rotating fluid
    - Hill chart
    - Generator
  - Governor
- Power Grid
Application

PRESSURE SURGE

- Joukowski - pressure surge (Жуковский)
  - sudden closure of a Valve
  - kinetic energy of fluid is converted in pressure
Application

OSCILLATING VALVE \( T_c = T_R \)

- Results
  - pressure and velocity in time at the valve
  - pressure and velocity in time in middle of the pipe

\[ \tau = \frac{A_{\text{valve}}}{A_{\text{pipe}}} \]
Application

OSCILLATING VALVE $T_\tau = 2T_R$

- Pressure and velocity variation along pipe
- $T_\tau = 2T_R$
Application

SURGE CHAMBER OSCILLATION

- surge chamber
  - reduces pressure fluctuations
  - hydraulic uncoupling of pressure tunnel from penstock
  - Improvement of control

- surge camber oscillation
  - Interaction of high frequent pressure waves and low frequent inertia waves
Application

SURGE CHAMBER OSCILLATION

- Model

\[ \Delta x = 137.2 \text{ [m]} \]
\[ \Delta t = 0.14263 \text{ [s]} \]
\[ a = 961.92 \text{ [m/s]} \]
\[ A_w = 79.0 \text{ [m}^2] \]
\[ D = 2.5 \text{ [m]} \]
\[ Q_0 = 12.25 \text{ [m}^3/\text{s]} \]
Application

SURGE CHAMBER OSCILLATION

- Results
  - Elevation of surge chamber water surface
  - Zustandsdiagramm
Application

SURGE CHAMBER OSCILLATION

- Results
  - pressure at the lower end of the penstock
  - pressure at the upper end of the penstock
Application

POWER PLANT

- Model
  - Simulation of plant, governor, grid and machine
  - Francis turbine
  - Isolated grid

\[ D_1 = 0.8 \text{ [m]} \]
\[ a_1 = 1230 \text{ [m/s]} \]
\[ D_2 = 0.6 \text{ [m]} \]
\[ a_2 = 1300 \text{ [m/s]} \]
\[ \Delta t = 8.6 \text{ [ms]} \]
\[ P_{T,\text{max}} = 1.0 \text{ [MW]} \]
Application

POWER PLANT

- start up
  - open loop control to overcome mass moment of inertia
  - synchronization
  - closed loop control by governor
Application

POWER PLANT

- load rejection
  - Sudden load rejection caused by failure in grid or generator
Thank you for your attention!
Implementation

- Method for calculating new time level
  - Initial solution
  - Constant velocity
  - Low pressure
Implementation

- Method for calculating new time level
  - valve closes suddenly
Implementation

- Method for calculating new time level
  - disturbance propagates upstream
  - half time step

\[ t = 1.5 \]
Implementation

- Method for calculating new time level
  - Next time level
  - calculate solution on intersection Point
Implementation

• Method for calculating new time level
Implementation

- Method for calculating new time level
Implementation

- Method for calculating new time level

\[ t = 3.0 \]