	$T \leftarrow D//$ unassigned clients
2: k	$\leftarrow 0$
	while $C \neq 0$ do
	$k \leftarrow k + 1$
5:	choose $j_k \in C$ that minimizes $v_j^* + \mathcal{C}_j^*$
	choose $i_k \in N(j_k)$ according to probability $x_{ij_k}$ .
7:	assign $j_k$ and all unassigned clients in $N^2(j_k)$ to $i_k$
8:	$C \leftarrow C - \{j_k\} - N^2(j_k)$

# Lemma 12 (Chernoff Bounds)

Let  $X_1, \ldots, X_n$  be *n* independent 0-1 random variables, not necessarily identically distributed. Then for  $X = \sum_{i=1}^{n} X_i$  and  $\mu = E[X], L \le \mu \le U, \text{ and } \delta > 0$ 

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U$$

and

$$\Pr[X \le (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L$$

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20.1 Chernoff Bounds

# Total assignment cost:

- Fix k; set  $j = j_k$ .
- Let  $\ell \in N^2(j)$  and h (one of) its neighbour(s) in N(j).
- If we assign a client  $\ell$  to the same facility as *i* we pay at most

$$\sum_{i} c_{ij} x_{ijk}^* + c_{hj} + c_{h\ell} \le C_j^* + v_j^* + v_\ell^* \le C_\ell^* + 2v_\ell^*$$

Summing this over all clients gives that the total assignment cost is at most

$$\sum_{j} C_{j}^{*} + \sum_{j} 2v_{j}^{*} \leq \sum_{j} C_{j}^{*} + 2\text{OPT}$$

Hence, it is at most 2OPT plus the total assignment cost in an optimum solution.

Adding the facility cost gives a 3-approximation.

# Lemma 13 For $0 \le \delta \le 1$ we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U \le e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$$

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20.1 Chernoff Bounds

# **Integer Multicommodity Flows**

- Given  $s_i$ - $t_i$  pairs in a graph.
- Connect each pair by a paths such that not too many path use any given edge.

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### Theorem 14

If  $W^* \ge c \ln n$  for some constant c, then with probability at least  $n^{-c/3}$  the total number of paths using any edge is at most  $W^* + \sqrt{cW^* \ln n}$ .

# **Integer Multicommodity Flows**

**Integer Multicommodity Flows** 

### **Randomized Rounding:**

For each i choose one path from the set  $\mathcal{P}_i$  at random according to the probability distribution given by the Linear Programming Solution.

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Let  $X_e^i$  be a random variable that indicates whether the path for  $s_i$ - $t_i$  uses edge e.

Then the number of paths using edge *e* is  $Y_e = \sum_i X_e^i$ .

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i: e \in p} x_p^* = \sum_{p: e \in P} x_p^* \le W^*$$

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# **Integer Multicommodity Flows**

Choose  $\delta = \sqrt{(c \ln n)/W^*}$ .

Then

$$\Pr[Y_e \ge (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

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# **Repetition: Primal Dual for Set Cover**

### Algorithm:

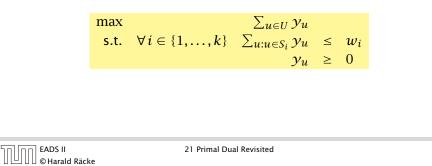
- Start with  $\gamma = 0$  (feasible dual solution). Start with x = 0 (integral primal solution that may be infeasible).
- $\blacktriangleright$  While x not feasible
  - Identify an element *e* that is not covered in current primal integral solution.
  - Increase dual variable  $y_e$  until a dual constraint becomes tight (maybe increase by 0!).
  - If this is the constraint for set  $S_i$  set  $x_i = 1$  (add this set to your solution).

# **Repetition: Primal Dual for Set Cover**

### Primal Relaxation:

min		$\sum_{i=1}^k w_i x_i$			
s.t.	$\forall u \in U$	$\sum_{i:u\in S_i} x_i$	$\geq$	1	
	$\forall i \in \{1,\ldots,k\}$	$x_i$	≥	0	

### **Dual Formulation:**



# **Repetition: Primal Dual for Set Cover** Analysis: For every set $S_i$ with $x_i = 1$ we have $\sum_{e \in S_i} y_e = w_j$ Hence our cost is $\sum_{j} w_{j} = \sum_{j} \sum_{e \in S_{i}} y_{e} = \sum_{e} |\{j : e \in S_{j}\}| \cdot y_{e} \le f \cdot \sum_{e} y_{e} \le f \cdot \text{OPT}$ EADS II © Harald Räcke

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