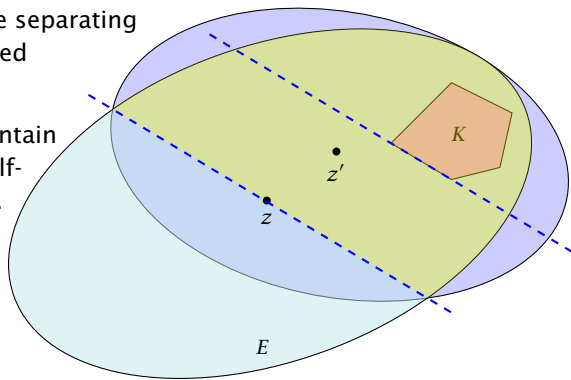


# Ellipsoid Method

- ▶ Let  $K$  be a convex set.
- ▶ Maintain ellipsoid  $E$  that is guaranteed to contain  $K$  provided that  $K$  is non-empty.
- ▶ If center  $z \in K$  STOP.
- ▶ Otw. find a hyperplane separating  $K$  from  $z$  (e.g. a violated constraint in the LP).
- ▶ Shift hyperplane to contain node  $z$ .  $H$  denotes half-space that contains  $K$ .
- ▶ Compute (smallest) ellipsoid  $E'$  that contains  $K \cap H$ .
- ▶ REPEAT



## Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ What if the polytop  $K$  is unbounded?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

### Definition 3

A mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $f(x) = Lx + t$ , where  $L$  is an invertible matrix is called an **affine transformation**.

## Definition 4

A ball in  $\mathbb{R}^n$  with center  $c$  and radius  $r$  is given by

$$\begin{aligned} B(c, r) &= \{x \mid (x - c)^t(x - c) \leq r^2\} \\ &= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\} \end{aligned}$$

$B(0, 1)$  is called the **unit ball**.

## Definition 5

An affine transformation of the unit ball is called an **ellipsoid**.

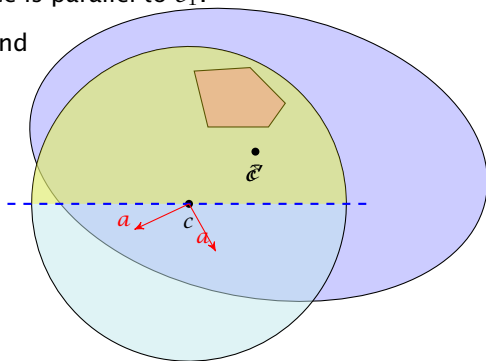
From  $f(x) = Lx + t$  follows  $x = L^{-1}(f(x) - t)$ .

$$\begin{aligned} f(B(0, 1)) &= \{f(x) \mid x \in B(0, 1)\} \\ &= \{y \in \mathbb{R}^n \mid L^{-1}(y - t) \in B(0, 1)\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^t L^{-1t} L^{-1}(y - t) \leq 1\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^t Q^{-1}(y - t) \leq 1\} \end{aligned}$$

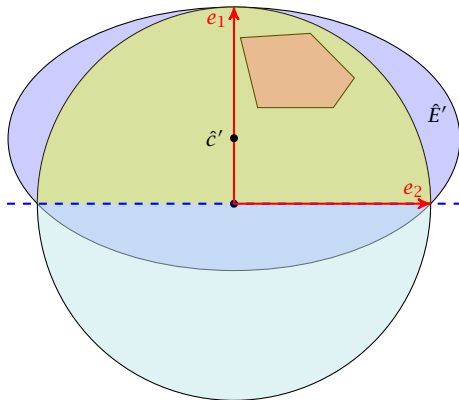
where  $Q = LL^t$  is an invertible matrix.

# How to Compute the New Ellipsoid

- ▶ Use  $f^{-1}$  (recall that  $f = Lx + t$  is the transformation function for the Ellipsoid) to rotate/distort the ellipsoid (back) into the unit ball.
- ▶ Use a rotation  $R^{-1}$  to rotate the unit ball such that the normal vector of the halfspace is parallel to  $e_1$ .
- ▶ Compute the new center  $\hat{c}'$  and the new matrix  $\hat{Q}'$  for this simplified setting.
- ▶ Use the transformations  $R$  and  $f$  to get the new center  $c'$  and the new matrix  $Q'$  for the original ellipsoid  $E$ .



# The Easy Case



- ▶ The new center lies on axis  $x_1$ . Hence,  $\hat{c}' = te_1$  for  $t > 0$ .
- ▶ The vectors  $e_1, e_2, \dots$  have to fulfill the ellipsoid constraint with equality. Hence  $(e_i - \hat{c}')^t \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ .

## The Easy Case

- ▶ To obtain the matrix  $\hat{Q}'^{-1}$  for our ellipsoid  $\hat{E}'$  note that  $\hat{E}'$  is **axis-parallel**.
- ▶ Let  $a$  denote the radius along the  $x_1$ -axis and let  $b$  denote the (common) radius for the other axes.
- ▶ The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function  $\hat{f}'(x) = \hat{L}'x$ ) to an axis-parallel ellipsoid with radius  $a$  in direction  $x_1$  and  $b$  in all other directions.



# The Easy Case

- ▶ As  $\hat{Q}' = \hat{L}'\hat{L}'^t$  the matrix  $\hat{Q}'^{-1}$  is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

# The Easy Case

- ▶  $(e_1 - \hat{c}')^t \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$  gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives  $(1-t)^2 = a^2$ .

## The Easy Case

- ▶ For  $i \neq 1$  the equation  $(e_i - \hat{c}')^t \hat{Q}'^{-1} (e_i - \hat{c}') = 1$  gives

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives  $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$ , and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

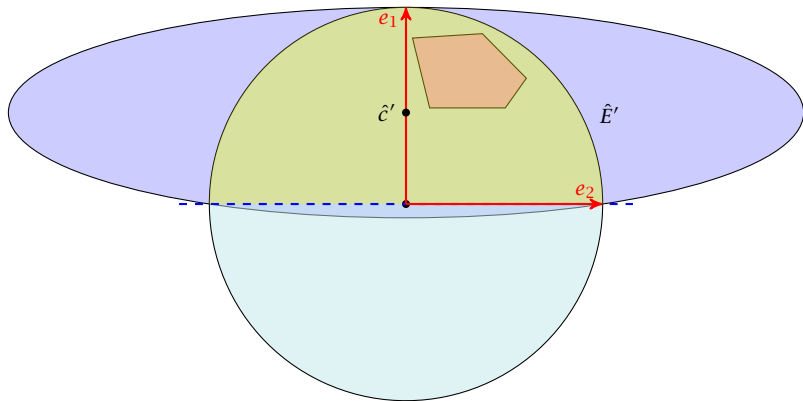
# Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

# The Easy Case

We still have many choices for  $t$ :



Choose  $t$  such that the volume of  $\hat{E}'$  is minimal!!!

# The Easy Case

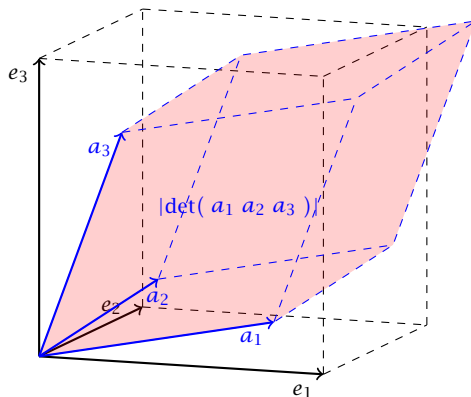
We want to choose  $t$  such that the volume of  $\hat{E}'$  is minimal.

## Lemma 6

Let  $L$  be an affine transformation and  $K \subseteq \mathbb{R}^n$ . Then

$$\text{vol}(L(K)) = |\det(L)| \cdot \text{vol}(K) .$$

# n-dimensional volume



## The Easy Case

- ▶ We want to choose  $t$  such that the volume of  $\hat{E}'$  is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0,1)) \cdot |\det(\hat{L}')| ,$$

where  $\hat{Q}' = \hat{L}'\hat{L}'^t$ .

- ▶ We have

$$\hat{L}'^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & \dots & 0 \\ 0 & \frac{1}{b} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b} \end{pmatrix} \text{ and } \hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- ▶ Note that  $a$  and  $b$  in the above equations depend on  $t$ , by the previous equations.



# The Easy Case

$$\begin{aligned}\text{vol}(\hat{E}') &= \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| \\ &= \text{vol}(B(0, 1)) \cdot ab^{n-1} \\ &= \text{vol}(B(0, 1)) \cdot (1-t) \cdot \left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1} \\ &= \text{vol}(B(0, 1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\end{aligned}$$

# The Easy Case

$$\begin{aligned}
 \frac{d \operatorname{vol}(\hat{E}')}{dt} &= \frac{d}{dt} \left( \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\
 &= \frac{1}{N^2} \cdot \left( \begin{array}{c} \text{derivative of numerator} \\ (-1) \cdot n(1-t)^{n-1} \cdot \frac{1-2t}{(\sqrt{1-2t})^{n-1}} \\ \text{denominator} \end{array} \right) \\
 &\quad \left( \begin{array}{c} \text{outer derivative} \\ (n-1)(\sqrt{1-2t})^{n-2} \cdot \frac{1}{2\sqrt{1-2t}} \cdot (-2) \cdot (1-t)^n \\ \text{inner derivative} \end{array} \right) \\
 &= \frac{1}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left( (n-1)(1-t) - n(1-2t) \right) \\
 &= \frac{1}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left( (n+1)t - 1 \right)
 \end{aligned}$$

# The Easy Case

- ▶ We obtain the minimum for  $t = \frac{1}{n+1}$ .
- ▶ For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}$$

To see the equation for  $b$ , observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{\left(1 - \frac{1}{n+1}\right)^2}{1 - \frac{2}{n+1}} = \frac{\left(\frac{n}{n+1}\right)^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}$$

## The Easy Case

Let  $\gamma_n = \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = ab^{n-1}$  be the ratio by which the volume changes:

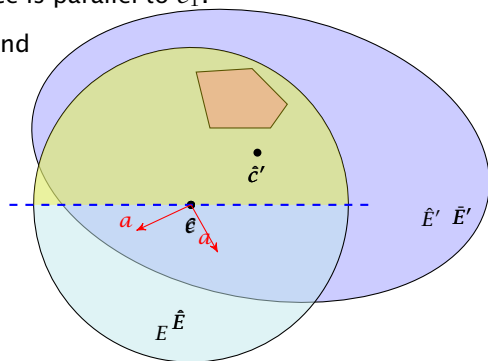
$$\begin{aligned}\gamma_n^2 &= \left(\frac{n}{n+1}\right)^2 \left(\frac{n^2}{n^2-1}\right)^{n-1} \\ &= \left(1 - \frac{1}{n+1}\right)^2 \left(1 + \frac{1}{(n-1)(n+1)}\right)^{n-1} \\ &\leq e^{-2\frac{1}{n+1}} \cdot e^{\frac{1}{n+1}} \\ &= e^{-\frac{1}{n+1}}\end{aligned}$$

where we used  $(1+x)^a \leq e^{ax}$  for  $x \in \mathbb{R}$  and  $a > 0$ .

This gives  $\gamma_n \leq e^{-\frac{1}{2(n+1)}}$ .

# How to Compute the New Ellipsoid

- ▶ Use  $f^{-1}$  (recall that  $f = Lx + t$  is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- ▶ Use a rotation  $R^{-1}$  to rotate the unit ball such that the normal vector of the halfspace is parallel to  $e_1$ .
- ▶ Compute the new center  $\hat{c}'$  and the new matrix  $\hat{Q}'$  for this simplified setting.
- ▶ Use the transformations  $R$  and  $f$  to get the new center  $c'$  and the new matrix  $Q'$  for the original ellipsoid  $E$ .



**Our progress is the same:**

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\geq \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0, 1))} = \frac{\text{vol}(\hat{E}')}{\text{vol}(\hat{E})} = \frac{\text{vol}(R(\hat{E}'))}{\text{vol}(R(\hat{E}))} \\ &= \frac{\text{vol}(\bar{E}')}{\text{vol}(\bar{E})} = \frac{\text{vol}(f(\bar{E}'))}{\text{vol}(f(\bar{E}))} = \frac{\text{vol}(E')}{\text{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function  $f(x) = Lx + t$  changes the volume by factor  $\det(L)$ .

# The Ellipsoid Algorithm

## How to Compute The New Parameters?

The transformation function of the (old) ellipsoid:  $f(x) = Lx + c$ ;

The halfspace to be intersected:  $H = \{x \mid a^t(x - c) \leq 0\}$ ;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^t(x - c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^t(f(y) - c) \leq 0\} \\ &= \{y \mid a^t(f(y) - c) \leq 0\} \\ &= \{y \mid a^t(Ly + c - c) \leq 0\} \\ &= \{y \mid (a^tL)y \leq 0\} \end{aligned}$$

This means  $\bar{a} = L^t a$ .

## The Ellipsoid Algorithm

After rotating back (applying  $R^{-1}$ ) the normal vector of the halfspace points in negative  $x_1$ -direction. Hence,

$$R^{-1}\left(\frac{L^t a}{\|L^t a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^t a}{\|L^t a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^t a}{\|L^t a\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^t a}{\|L^t a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^t Q a}} \end{aligned}$$



For computing the matrix  $Q'$  of the new ellipsoid we assume in the following that  $\hat{E}'$ ,  $\bar{E}'$  and  $E'$  refer to the ellipsoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left( I - \frac{2}{n + 1} e_1 e_1^t \right)$$

because for  $a = n/n+1$  and  $b = n/\sqrt{n^2-1}$

$$\begin{aligned} b^2 - b^2 \frac{2}{n + 1} &= \frac{n^2}{n^2 - 1} - \frac{2n^2}{(n - 1)(n + 1)^2} \\ &= \frac{n^2(n + 1) - 2n^2}{(n - 1)(n + 1)^2} = \frac{n^2(n - 1)}{(n - 1)(n + 1)^2} = a^2 \end{aligned}$$

## 9 The Ellipsoid Algorithm

$$\begin{aligned}\tilde{E}' &= R(\hat{E}') \\ &= \{R(\mathbf{x}) \mid \mathbf{x}^t \hat{Q}'^{-1} \mathbf{x} \leq 1\} \\ &= \{\mathbf{y} \mid (R^{-1}\mathbf{y})^t \hat{Q}'^{-1} R^{-1}\mathbf{y} \leq 1\} \\ &= \{\mathbf{y} \mid \mathbf{y}^t (R^t)^{-1} \hat{Q}'^{-1} R^{-1}\mathbf{y} \leq 1\} \\ &= \{\mathbf{y} \mid \mathbf{y}^t \underbrace{(R\hat{Q}'R^t)^{-1}}_{\tilde{Q}'} \mathbf{y} \leq 1\}\end{aligned}$$

## 9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^t \\ &= R \cdot \frac{n^2}{n^2-1} \left( I - \frac{2}{n+1} e_1 e_1^t \right) \cdot R^t \\ &= \frac{n^2}{n^2-1} \left( R \cdot R^t - \frac{2}{n+1} (Re_1)(Re_1)^t \right) \\ &= \frac{n^2}{n^2-1} \left( I - \frac{2}{n+1} \frac{L^t a a^t L}{\|L^t a\|^2} \right)\end{aligned}$$

## 9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^t \bar{Q}'^{-1} x \leq 1\} \\ &= \{y \mid (L^{-1}y)^t \bar{Q}'^{-1} L^{-1}y \leq 1\} \\ &= \{y \mid y^t (L^t)^{-1} \bar{Q}'^{-1} L^{-1}y \leq 1\} \\ &= \{y \mid y^t \underbrace{(L\bar{Q}'L^t)^{-1}}_{Q'} y \leq 1\} \end{aligned}$$

## 9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}Q' &= L\bar{Q}'L^t \\ &= L \cdot \frac{n^2}{n^2-1} \left( I - \frac{2}{n+1} \frac{L^t a a^t L}{a^t Q a} \right) \cdot L^t \\ &= \frac{n^2}{n^2-1} \left( Q - \frac{2}{n+1} \frac{Q a a^t Q}{a^t Q a} \right)\end{aligned}$$

# Incomplete Algorithm

## Algorithm 1 ellipsoid-algorithm

- 1: **input:** point  $c \in \mathbb{R}^n$ , convex set  $K \subseteq \mathbb{R}^n$
- 2: **output:** point  $x \in K$  or “ $K$  is empty”
- 3:  $Q \leftarrow ???$
- 4: **repeat**
- 5:     **if**  $c \in K$  **then return**  $c$
- 6:     **else**
- 7:         choose a violated hyperplane  $a$
- 8:         
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^t Q a}}$$
- 9:         
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left( Q - \frac{2}{n+1} \frac{Qaa^t Q}{a^t Q a} \right)$$
- 10:     **endif**
- 11: **until**  $???$
- 12: **return** “ $K$  is empty”

## Repeat: Size of basic solutions

### Lemma 7

Let  $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$  be a bounded polytop. Let  $\langle a_{\max} \rangle$  be the maximum encoding length of an entry in  $A$ . Then every entry  $x_j$  in a basic solution fulfills  $|x_j| = \frac{D_j}{D}$  with  $D_j, D \leq 2^{2n\langle a_{\max} \rangle + n \log_2 n}$ .

In the following we use  $\delta := 2^{n\langle a_{\max} \rangle + n \log_2 n}$ .

Note that here we have  $P = \{x \mid Ax \leq b\}$ . The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.



## Repeat: Size of basic solutions

**Proof:**

Let  $\bar{A} = \begin{bmatrix} A & \\ -A & I_m \end{bmatrix}$ ,  $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$ , be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices  $\bar{A}_B$  and  $\bar{M}_j$  (matrix obt. when replacing the  $j$ -th column of  $\bar{A}_B$  by  $\bar{b}$ ) can become at most

$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^n \\ &\leq (\sqrt{n} \cdot 2^{\langle a_{\max} \rangle})^n \leq 2^{n \langle a_{\max} \rangle + n \log_2 n}, \end{aligned}$$

where  $\vec{\ell}_{\max}$  is the longest column-vector that can be obtained after deleting all but  $n$  rows and columns from  $\bar{A}$ .

This holds because columns from  $I_m$  selected when going from  $\bar{A}$  to  $\bar{A}_B$  do not increase the determinant. Only the at most  $n$  columns from matrices  $A$  and  $-A$  that  $\bar{A}$  consists of contribute.

## How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop  $P$  is bounded.

In this case every entry  $x_i$  in a basic solution fulfills  $|x_i| \leq \delta$ .

Hence,  $P$  is contained in the cube  $-\delta \leq x_i \leq \delta$ .

A vector in this cube has at most distance  $R := \sqrt{n}\delta$  from the origin.

Starting with the ball  $E_0 := B(0, R)$  ensures that  $P$  is completely contained in the initial ellipsoid. This ellipsoid has volume at most  $R^n B(0, 1) \leq (n\delta)^n B(0, 1)$ .

## When can we terminate?

Let  $P := \{x \mid Ax \leq b\}$  with  $A \in \mathbb{Z}$  and  $b \in \mathbb{Z}$  be a bounded polytop. Let  $\langle a_{\max} \rangle$  be the encoding length of the largest entry in  $A$  or  $b$ .

Consider the following polytope

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where  $\lambda = \delta^2 + 1$ .

## Lemma 8

*$P_\lambda$  is feasible if and only if  $P$  is feasible.*

$\Leftarrow$ : obvious!

$\Rightarrow$ :

Consider the polytop

$$\bar{P} = \left\{ x \mid \begin{bmatrix} A & \\ -A & I_m \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}; x \geq 0 \right\}$$

and

$$\bar{P}_\lambda = \left\{ x \mid \begin{bmatrix} A & \\ -A & I_m \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0 \right\} .$$

$P$  is feasible if and only if  $\bar{P}$  is feasible, and  $P_\lambda$  feasible if and only if  $\bar{P}_\lambda$  feasible.

$\bar{P}_\lambda$  is bounded since  $P_\lambda$  and  $P$  are bounded.

Let  $\bar{A} = \begin{bmatrix} A & \\ -A & I_m \end{bmatrix}$ , and  $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$ .

$\bar{P}_\lambda$  feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1} \bar{b} + \frac{1}{\lambda} \bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other  $x$ -values are zero)

The only reason that this basic feasible solution is not feasible for  $\bar{P}$  is that one of the basic variables becomes negative.

Hence, there exists  $i$  with

$$(\bar{A}_B^{-1} \bar{b})_i < 0 \leq (\bar{A}_B^{-1} \bar{b})_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i$$

By Cramers rule we get

$$(\bar{A}_B^{-1}\bar{\mathbf{b}})_i < 0 \quad \Rightarrow \quad (\bar{A}_B^{-1}\bar{\mathbf{b}})_i \leq -\frac{1}{\det(\bar{A}_B)}$$

and

$$(\bar{A}_B^{-1}\vec{\mathbf{1}})_i \leq \det(\bar{M}_j) ,$$

where  $\bar{M}_j$  is obtained by replacing the  $j$ -th column of  $\bar{A}_B$  by  $\vec{\mathbf{1}}$ .

However, we showed that the determinants of  $\bar{A}_B$  and  $\bar{M}_j$  can become at most  $\delta$ .

Since, we chose  $\lambda = \delta^2 + 1$  this gives a contradiction.

## Lemma 9

If  $P_\lambda$  is feasible then it contains a ball of radius  $r := 1/\delta^3$ . This has a volume of at least  $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$ .

### Proof:

If  $P_\lambda$  feasible then also  $P$ . Let  $x$  be feasible for  $P$ . This means  $Ax \leq b$ .

Let  $\vec{\ell}$  with  $\|\vec{\ell}\| \leq r$ . Then

$$\begin{aligned}(A(x + \vec{\ell}))_i &= (Ax)_i + (A\vec{\ell})_i \leq b_i + A_i \vec{\ell} \\ &\leq b_i + \|A_i\| \cdot \|\vec{\ell}\| \leq b_i + \sqrt{n} \cdot 2^{\langle a_{\max} \rangle} \cdot r \\ &\leq b_i + \frac{\sqrt{n} \cdot 2^{\langle a_{\max} \rangle}}{\delta^3} \leq b_i + \frac{1}{\delta^2 + 1} \leq b_i + \frac{1}{\lambda}\end{aligned}$$

Hence,  $x + \vec{\ell}$  is feasible for  $P_\lambda$  which proves the lemma.



How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \text{vol}(B(0, R)) < \text{vol}(B(0, r))$$

Hence,

$$\begin{aligned} i &> 2(n+1) \ln\left(\frac{\text{vol}(B(0, R))}{\text{vol}(B(0, r))}\right) \\ &= 2(n+1) \ln(n^n \delta^n \cdot \delta^{3n}) \\ &= 8n(n+1) \ln(\delta) + 2(n+1)n \ln(n) \\ &= \mathcal{O}(\text{poly}(n, \langle a_{\max} \rangle)) \end{aligned}$$

## Algorithm 1 ellipsoid-algorithm

- 1: **input:** point  $c \in \mathbb{R}^n$ , convex set  $K \subseteq \mathbb{R}^n$ , radii  $R$  and  $r$
- 2:           with  $K \subseteq B(0, R)$ , and  $B(x, r) \subseteq K$  for some  $x$
- 3: **output:** point  $x \in K$  or “ $K$  is empty”
- 4:  $Q \leftarrow \text{diag}(R^2, \dots, R^2)$  // i.e.,  $L = \text{diag}(R, \dots, R)$
- 5:  $c \leftarrow 0$
- 6: **repeat**
- 7:     **if**  $c \in K$  **then return**  $c$
- 8:     **else**
- 9:         choose a violated hyperplane  $a$
- 10:         
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^t Q a}}$$
- 11:         
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left( Q - \frac{2}{n+1} \frac{Q a a^t Q}{a^t Q a} \right)$$
- 12:     **endif**
- 13: **until**  $\det(Q) \leq r^{2n}$  // i.e.,  $\det(L) \leq r^n$
- 14: **return** “ $K$  is empty”

## Separation Oracle:

Let  $K \subseteq \mathbb{R}^n$  be a convex set. A separation oracle for  $K$  is an algorithm  $A$  that gets as input a point  $x \in \mathbb{R}^n$  and either

- ▶ certifies that  $x \in K$ ,
- ▶ or finds a hyperplane separating  $x$  from  $K$ .

We will usually assume that  $A$  is a polynomial-time algorithm.

In order to find a point in  $K$  we need

- ▶ a guarantee that a ball of radius  $r$  is contained in  $K$ ,
- ▶ an initial ball  $B(c, R)$  with radius  $R$  that contains  $K$ ,
- ▶ a separation oracle for  $K$ .

The Ellipsoid algorithm requires  $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$  iterations. Each iteration is polytime for a polynomial-time Separation oracle.