## 17 Rounding Data + Dynamic Programming

### Knapsack:

Given a set of items  $\{1, ..., n\}$ , where the *i*-th item has weight  $w_i \in \mathbb{N}$  and profit  $p_i \in \mathbb{N}$ , and given a threshold W. Find a subset  $I \subseteq \{1, ..., n\}$  of items of total weight at most W such that the profit is maximized (we can assume each  $w_i \leq W$ ).

	max s.t.	$\forall i \in \{1, \dots, n\}$	$\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} w_i x_i}$	≤ ∈	W {0,1}
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## 17 Rounding Data + Dynamic Programming

#### **Definition 2**

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An algorithm is said to have pseudo-polynomial running time if the running time is polynomial when the numerical part of the input is encoded in unary.

17.1 Knapsack

# 17 Rounding Data + Dynamic Programming

Algorithm 1 Knapsack
1: $A(1) \leftarrow [(0,0), (p_1, w_1)]$
2: for $j \leftarrow 2$ to $n$ do
3: $A(j) \leftarrow A(j-1)$
4: for each $(p, w) \in A(j-1)$ do
5: <b>if</b> $w + w_j \le W$ then
6: $add (p + p_j, w + w_j) \text{ to } A(j)$
7: remove dominated pairs from $A(j)$
8: return $\max_{(p,w)\in A(n)} p$

The running time is  $\mathcal{O}(n \cdot \min\{W, P\})$ , where  $P = \sum_i p_i$  is the total profit of all items. This is only pseudo-polynomial.

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17.1 Knapsack
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## 17 Rounding Data + Dynamic Programming

- Let *M* be the maximum profit of an element.
- Set  $\mu := \epsilon M/n$ .
- Set  $p'_i := \lfloor p_i / \mu \rfloor$  for all *i*.
- Run the dynamic programming algorithm on this revised instance.

Running time is at most

$$\mathcal{O}(nP') = \mathcal{O}(n\sum_{i} p'_{i}) = \mathcal{O}(n\sum_{i} \lfloor \frac{p_{i}}{\epsilon M/n} \rfloor) \le \mathcal{O}(\frac{n^{3}}{\epsilon})$$

17.1 Knapsack

## 17 Rounding Data + Dynamic Programming

Let S be the set of items returned by the algorithm, and let O be an optimum set of items.

$$\sum_{i \in S} p_i \ge \mu \sum_{i \in S} p'_i$$

$$\ge \mu \sum_{i \in O} p'_i$$

$$\ge \sum_{i \in O} p_i - |O|\mu$$

$$\ge \sum_{i \in O} p_i - n\mu$$

$$= \sum_{i \in O} p_i - \epsilon M$$

$$\ge (1 - \epsilon) \text{OPT} .$$
17.1 Knapsack

# **17.2 Scheduling Revisited**

Partition the input into long jobs and short jobs.

A job j is called short if

$$p_j \le \frac{1}{km} \sum_i p_i$$

### Idea:

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- 1. Find the optimum Makespan for the long jobs by brute force.
- 2. Then use the list scheduling algorithm for the short jobs, always assigning the next job to the least loaded machine.

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## **Scheduling Revisited**

The previous analysis of the scheduling algorithm gave a makespan of

$$\frac{1}{m}\sum_{j\neq\ell}p_j+p_\ell$$

where  $\ell$  is the last job to complete.

Together with the observation that if each  $p_i \ge \frac{1}{3}C_{\text{max}}^*$  then LPT is optimal this gave a 4/3-approximation.

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17.2 Scheduling Revisited

We still have the inequality

$$\frac{1}{m}\sum_{j\neq\ell}p_j+p_\ell$$

where  $\ell$  is the last job (this only requires that all machines are busy before time  $S_{\ell}$ ).

If  $\ell$  is a long job, then the schedule must be optimal, as it consists of an optimal schedule of long jobs plus a schedule for short jobs.

If  $\ell$  is a short job its length is at most

$$p_\ell \leq \sum_j p_j / (mk)$$

which is at most  $C^*_{\max}/k$ .

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Hence we get a schedule of length at most

 $(1+\frac{1}{k})C_{\max}^*$ 

There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most  $m^{km}$ , which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

### **Theorem 3**

The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

We choose  $k = \lceil \frac{1}{\epsilon} \rceil$ .

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17.2 Scheduling Revisited

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- We round all long jobs down to multiples of  $T/k^2$ .
- For these rounded sizes we first find an optimal schedule.
- If this schedule does not have length at most T we conclude that also the original sizes don't allow such a schedule.
- If we have a good schedule we extend it by adding the short jobs according to the LPT rule.

How to get rid of the requirement that m is constant?

We first design an algorithm that works as follows: On input of *T* it either finds a schedule of length  $(1 + \frac{1}{k})T$  or certifies that no schedule of length at most *T* exists (assume  $T \ge \frac{1}{m} \sum_{j} p_{j}$ ).

We partition the jobs into long jobs and short jobs:

- A job is long if its size is larger than T/k.
- Otw. it is a short job.

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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most T.

There can be at most k (long) jobs assigned to a machine as otw. their rounded sizes would add up to more than T (note that the rounded size of a long job is at least T/k).

Since, jobs had been rounded to multiples of  $T/k^2$  going from rounded sizes to original sizes gives that the Makespan is at most During the second phase there always must exist a machine with load at most T, since T is larger than the average load. Assigning the current (short) job to such a machine gives that the new load is at most

$$T + \frac{T}{k} \le (1 + \frac{1}{k})T$$

	17.2 Scheduling Revisited	222	٦
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Let  $OPT(n_1, ..., n_{k^2})$  be the number of machines that are required to schedule input vector  $(n_1, ..., n_{k^2})$  with Makespan at most T.

### If $OPT(n_1, \ldots, n_{k^2}) \le m$ we can schedule the input.

#### We have

 $OPT(n_1,...,n_{k^2})$ 

$$= \begin{cases} 0 & (n_1, \dots, n_{k^2}) = 0\\ 1 + \min_{(s_1, \dots, s_{k^2}) \in C} \operatorname{OPT}(n_1 - s_1, \dots, n_{k^2} - s_{k^2}) & (n_1, \dots, n_{k^2}) \ge 0\\ \infty & \text{otw.} \end{cases}$$

where *C* is the set of all configurations.

Hence, the running time is roughly  $(k + 1)^{k^2} n^{k^2} \approx (nk)^{k^2}$ .

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**Running Time for scheduling large jobs:** There should not be a job with rounded size more than T as otw. the problem becomes trivial.

Hence, any large job has rounded size of  $\frac{i}{k^2}T$  for  $i \in \{k, ..., k^2\}$ . Therefore the number of different inputs is at most  $n^{k^2}$ (described by a vector of length  $k^2$  where, the *i*-th entry describes the number of jobs of size  $\frac{i}{k^2}T$ ). This is polynomial.

The schedule/configuration of a particular machine x can be described by a vector of length  $k^2$  where the *i*-th entry describes the number of jobs of rounded size  $\frac{i}{k^2}T$  assigned to x. There are only  $(k + 1)^{k^2}$  different vectors.

This means there are a constant number of different machine configurations.

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17.2 Scheduling Revisited

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We can turn this into a PTAS by choosing  $k = \lceil 1/\epsilon \rceil$  and using binary search. This gives a running time that is exponential in  $1/\epsilon$ .

#### Can we do better?

Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

### Theorem 4

There is no FPTAS for problems that are strongly NP-hard.

## **More General**

Let  $OPT(n_1, ..., n_A)$  be the number of machines that are required to schedule input vector  $(n_1, ..., n_A)$  with Makespan at most T (*A*: number of different sizes).

If  $OPT(n_1, ..., n_A) \le m$  we can schedule the input.

 $OPT(n_1,\ldots,n_A)$ 

 $= \begin{cases} 0 & (n_1, \dots, n_A) = 0\\ 1 + \min_{(s_1, \dots, s_A) \in C} OPT(n_1 - s_1, \dots, n_A - s_A) & (n_1, \dots, n_A) \ge 0\\ \infty & \text{otw.} \end{cases}$ 

where C is the set of all configurations.

 $|C| \le (B+1)^A$ , where *B* is the number of jobs that possibly can fit on the same machine.

The running time is then  $O((B + 1)^A n^A)$  because the dynamic programming table has just  $n^A$  entries.

## **Bin Packing**

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### Proof

▶ In the partition problem we are given positive integers  $b_1, ..., b_n$  with  $B = \sum_i b_i$  even. Can we partition the integers into two sets *S* and *T* s.t.

$$\sum_{i\in S} b_i = \sum_{i\in T} b_i \quad ?$$

- We can solve this problem by setting  $s_i := 2b_i/B$  and asking whether we can pack the resulting items into 2 bins or not.
- A ρ-approximation algorithm with ρ < 3/2 cannot output 3 or more bins when 2 are optimal.

17.3 Bin Packing

• Hence, such an algorithm can solve Partition.

# **Bin Packing**

Given *n* items with sizes  $s_1, \ldots, s_n$  where

 $1 > s_1 \geq \cdots \geq s_n > 0$ .

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

### Theorem 5

There is no  $\rho$ -approximation for Bin Packing with  $\rho < 3/2$  unless P = NP.

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17.3 Bin Packing

## Bin Packing

### **Definition 6**

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms  $\{A_{\epsilon}\}$  along with a constant c such that  $A_{\epsilon}$  returns a solution of value at most  $(1 + \epsilon)$ OPT + c for minimization problems.

- Note that for Set Cover or for Knapsack it makes no sense to differentiate between the notion of a PTAS or an APTAS because of scaling.
- However, we will develop an APTAS for Bin Packing.

## **Bin Packing**

Again we can differentiate between small and large items.

### Lemma 7

Any packing of items of size at most  $\gamma$  into  $\ell$  bins can be extended to a packing of all items into  $\max\{\ell, \frac{1}{1-\gamma}SIZE(I) + 1\}$  bins, where  $SIZE(I) = \sum_i s_i$  is the sum of all item sizes.

- ► If after Greedy we use more than  $\ell$  bins, all bins (apart from the last) must be full to at least  $1 \gamma$ .
- Hence, r(1 − γ) ≤ SIZE(I) where r is the number of nearly-full bins.
- This gives the lemma.

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n Packing

# **Bin Packing**

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### Linear Grouping:

Generate an instance I' (for large items) as follows.

- Order large items according to size.
- Let the first k items belong to group 1; the following k items belong to group 2; etc.
- Delete items in the first group;
- Round items in the remaining groups to the size of the largest item in the group.

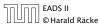
17.3 Bin Packing

Choose  $\gamma = \epsilon/2$ . Then we either use  $\ell$  bins or at most

$$\frac{1}{1 - \epsilon/2} \cdot \text{OPT} + 1 \le (1 + \epsilon) \cdot \text{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.



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17.3 Bin Packing

#### Lemma 8

 $OPT(I') \le OPT(I) \le OPT(I') + k$ 

### Proof 1:

- Any bin packing for *I* gives a bin packing for *I* as follows.
- Pack the items of group 2, where in the packing for *I* the items for group 1 have been packed;
- Pack the items of groups 3, where in the packing for *I* the items for group 2 have been packed;
- ▶ ...

# Lemma 9

 $OPT(I') \le OPT(I) \le OPT(I') + k$ 

### Proof 2:

- Any bin packing for I' gives a bin packing for I as follows.
- Pack the items of group 1 into k new bins;
- Pack the items of groups 2, where in the packing for I' the items for group 2 have been packed;

▶ ...

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## Can we do better?

In the following we show how to obtain a solution where the number of bins is only

$$OPT(I) + O(log^2(SIZE(I)))$$

Note that this is usually better than a guarantee of

$$(1 + \epsilon)$$
OPT $(I) + 1$ 

Assume that our instance does not contain pieces smaller than  $\epsilon/2$ . Then SIZE(I)  $\geq \epsilon n/2$ .

We set  $k = \lfloor \epsilon \text{SIZE}(I) \rfloor$ .

Then  $n/k \le 2n/\lfloor \epsilon^2 n/2 \rfloor \le 4/\epsilon^2$  (here we used  $\lfloor \alpha \rfloor \ge \alpha/2$  for  $\alpha \ge 1$ ).

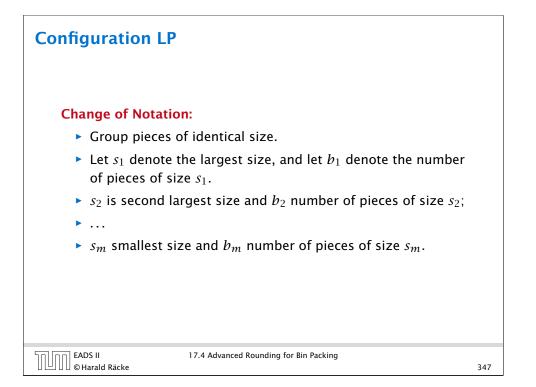
Hence, after grouping we have a constant number of piece sizes  $(4/\epsilon^2)$  and at most a constant number  $(2/\epsilon)$  can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

cost (for large items) at most

 $OPT(I') + k \le OPT(I) + \epsilon SIZE(I) \le (1 + \epsilon)OPT(I)$ 

• running time  $\mathcal{O}((\frac{2}{\epsilon}n)^{4/\epsilon^2})$ .



17.4 Advanced Rounding for Bin Packing

# **Configuration LP**

A possible packing of a bin can be described by an *m*-tuple  $(t_1, \ldots, t_m)$ , where  $t_i$  describes the number of pieces of size  $s_i$ . Clearly,

 $\sum_i t_i \cdot s_i \leq 1$  .

We call a vector that fulfills the above constraint a configuration.

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How to solve this	LP?	
later		
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## **Configuration LP**

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Let N be the number of configurations (exponential).

Let  $T_1, \ldots, T_N$  be the sequence of all possible configurations (a configuration  $T_j$  has  $T_{ji}$  pieces of size  $s_i$ ).

min		$\sum_{i=1}^{N} x_i$			
s.t.	$\forall i \in \{1 \dots m\}$	$\sum_{j=1}^{N} T_{ji} x_j$	≥	$b_i$	
	$\forall j \in \{1, \dots, N\}$	$x_j$	≥	0	
	$\forall j \in \{1, \dots, N\}$	$x_j$	integral		
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We can assume that each item has size at least 1/SIZE(I).

## **Harmonic Grouping**

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G<sub>1</sub> is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for G<sub>2</sub>,...,G<sub>r-1</sub>.
- Only the size of items in the last group G<sub>r</sub> may sum up to less than 2.

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### Lemma 10

The number of different sizes in I' is at most SIZE(I)/2.

- Each group that survives (recall that G<sub>1</sub> and G<sub>r</sub> are deleted) has total size at least 2.
- Hence, the number of surviving groups is at most SIZE(I)/2.
- All items in a group have the same size in I'.

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# Harmonic Grouping

From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
- Delete all items from group  $G_1$  and  $G_r$ .
- For groups  $G_2, \ldots, G_{r-1}$  delete  $n_i n_{i-1}$  items.
- Observe that  $n_i \ge n_{i-1}$ .

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### Lemma 11

The total size of deleted items is at most  $O(\log(SIZE(I)))$ .

- ► The total size of items in G<sub>1</sub> and G<sub>r</sub> is at most 6 as a group has total size at most 3.
- Consider a group  $G_i$  that has strictly more items than  $G_{i-1}$ .
- It discards  $n_i n_{i-1}$  pieces of total size at most

$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the smallest piece has size at most  $3/n_i$ .

Summing over all *i* that have n<sub>i</sub> > n<sub>i-1</sub> gives a bound of at most

$$\sum_{j=1}^{l_{r-1}} \frac{3}{j} \le \mathcal{O}(\log(\text{SIZE}(I))) \quad .$$

(note that  $n_r \leq \text{SIZE}(I)$  since we assume that the size of each item is at least 1/SIZE(I)).

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### Algorithm 1 BinPack

- 1: **if** SIZE(I) < 10 **then**
- 2: pack remaining items greedily
- 3: Apply harmonic grouping to create instance I'; pack discarded items in at most  $O(\log(SIZE(I)))$  bins.
- 4: Let x be optimal solution to configuration LP
- 5: Pack  $\lfloor x_j \rfloor$  bins in configuration  $T_j$  for all j; call the packed instance  $I_1$ .
- 6: Let  $I_2$  be remaining pieces from I'
- 7: Pack  $I_2$  via BinPack $(I_2)$

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## Analysis

Each level of the recursion partitions pieces into three types

- 1. Pieces discarded at this level.
- **2.** Pieces scheduled because they are in  $I_1$ .
- **3.** Pieces in  $I_2$  are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most  $\mbox{OPT}_{\mbox{LP}}$  many bins.

Pieces of type 1 are packed into at most

### $\mathcal{O}(\log(\text{SIZE}(I))) \cdot L$

many bins where L is the number of recursion levels.

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# Analysis

### $OPT_{LP}(I_1) + OPT_{LP}(I_2) \le OPT_{LP}(I') \le OPT_{LP}(I)$

### Proof:

- ► Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, OPT<sub>LP</sub>(I') ≤ OPT<sub>LP</sub>(I)
- $\lfloor x_j \rfloor$  is feasible solution for  $I_1$  (even integral).
- $x_j \lfloor x_j \rfloor$  is feasible solution for  $I_2$ .

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# Analysis

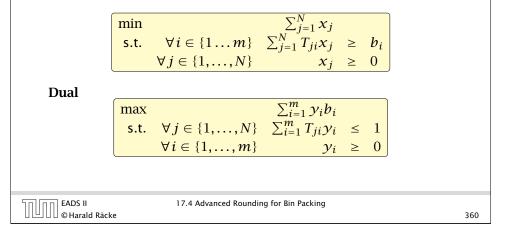
We can show that  $SIZE(I_2) \le SIZE(I)/2$ . Hence, the number of recursion levels is only  $O(\log(SIZE(I_{original})))$  in total.

- ► The number of non-zero entries in the solution to the configuration LP for I' is at most the number of constraints, which is the number of different sizes (≤ SIZE(I)/2).
- ► The total size of items in  $I_2$  can be at most  $\sum_{j=1}^{N} x_j \lfloor x_j \rfloor$  which is at most the number of non-zero entries in the solution to the configuration LP.

## How to solve the LP?

Let  $T_1, \ldots, T_N$  be the sequence of all possible configurations (a configuration  $T_j$  has  $T_{ji}$  pieces of size  $s_i$ ). In total we have  $b_i$  pieces of size  $s_i$ .

### Primal



## **Separation Oracle**

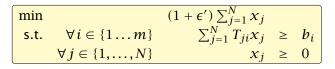
We have FPTAS for Knapsack. This means if a constraint is violated with  $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$  we find it, since we can obtain at least  $(1 - \epsilon)$  of the optimal profit.

The solution we get is feasible for:

Dual'

 $\begin{array}{|c|c|c|c|c|c|} \hline \max & & \sum_{i=1}^{m} y_i b_i \\ \text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 + \epsilon' \\ & \forall i \in \{1, \dots, m\} & & y_i & \geq & 0 \end{array}$ 

Primal'



## **Separation Oracle**

Suppose that I am given variable assignment y for the dual.

### How do I find a violated constraint?

I have to find a configuration  $T_j = (T_{j1}, \ldots, T_{jm})$  that

m

is feasible, i.e.,

$$\sum_{i=1}^m T_{ji} \cdot s_i \le 1 \quad ,$$

and has a large profit

$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.

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## **Separation Oracle**

If the value of the computed dual solution (which may be infeasible) is z then

$$OPT \le z \le (1 + \epsilon')OPT$$

### How do we get good primal solution (not just the value)?

- The constraints used when computing z certify that the solution is feasible for DUAL'.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for DUAL'.
- ► Let DUAL'' be DUAL without unused constraints.
- The dual to DUAL" is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
- The optimum value for PRIMAL'' is at most  $(1 + \epsilon')$ OPT.
- > We can compute the corresponding solution in polytime.

This gives that overall we need at most

$$(1 + \epsilon')$$
OPT<sub>LP</sub> $(I) + O(\log^2(SIZE(I)))$ 

bins.

We can choose  $\epsilon' = \frac{1}{OPT}$  as  $OPT \le \#$ items and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.

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