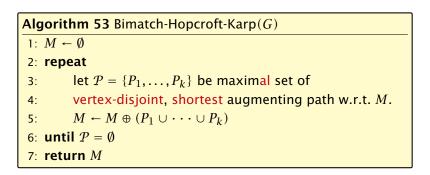
A Fast Matching Algorithm



We call one iteration of the repeat-loop a phase of the algorithm.



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Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

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- ► Let $P_1, ..., P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. *M* (let $\ell = |P_i|$).
- $M' \cong M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k.$
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- ► The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
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P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

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- Hence, $|A| \leq k\ell + |P| 1$.
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If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.

The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.



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Lemma 7

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

- ► After iteration $\lfloor \sqrt{|V|} \rfloor$ the length of a shortest augmenting path must be at least $\lfloor \sqrt{|V|} \rfloor + 1 \ge \sqrt{|V|}$.
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Lemma 8

One phase of the Hopcroft-Karp algorithm can be implemented in time O(m).

Do a breadth first search starting at all free vertices in the left side L.

(alternatively add a super-startnode; connect it to all free vertices in L and start breadth first search from there)

The search stops when reaching a free vertex. However, the current level of the BFS tree is still finished in order to find a set *F* of free vertices (on the right side) that can be reached via shortest augmenting paths.



- Then a maximal set of shortest path from the leftmost layer of the tree construction to nodes in F needs to be computed.
- Any such path must visit the layers of the BFS-tree from left to right.
- To go from an odd layer to an even layer it must use a matching edge.
- To go from an even layer to an odd layer edge it can use edges in the BFS-tree or edges that have been ignored during BFS-tree construction.
- We direct all edges btw. an even node in some layer ℓ to an odd node in layer $\ell + 1$ from left to right.
- A DFS search in the resulting graph gives us a maximal set of vertex disjoint path from left to right in the resulting graph.



