A crucial ingredient for the design and analysis of approximation algorithms is a technique to obtain an upper bound (for maximization problems) or a lower bound (for minimization problems).

Therefore Linear Programs or Integer Linear Programs play a vital role in the design of many approximation algorithms.

### **Definition 2**

An Integer Linear Program or Integer Program is a Linear Program in which all variables are required to be integral.

### **Definition 3**

A Mixed Integer Program is a Linear Program in which a subset of the variables are required to be integral.

Many important combinatorial optimization problems can be formulated in the form of an Integer Program.

Note that solving Integer Programs in general is NP-complete!

### **Set Cover**

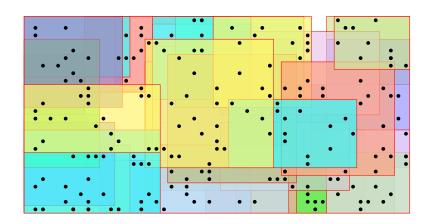
Given a ground set U, a collection of subsets  $S_1, \ldots, S_k \subseteq U$ , where the i-th subset  $S_i$  has weight/cost  $w_i$ . Find a collection  $I \subseteq \{1, \ldots, k\}$  such that

$$\forall u \in U \exists i \in I : u \in S_i$$
 (every element is covered)

and

$$\sum_{i \in I} w_i$$
 is minimized.

## **Set Cover**



## **IP-Formulation of Set Cover**

min		$\sum_i w_i x_i$		
s.t.	$\forall u \in U$	$\sum_{i:u\in S_i} x_i$	≥	1
	$\forall i \in \{1, \ldots, k\}$	$x_i$	≥	0
	$\forall i \in \{1, \ldots, k\}$	$x_i$	integral	

### **Vertex Cover**

Given a graph G=(V,E) and a weight  $w_v$  for every node. Find a vertex subset  $S\subseteq V$  of minimum weight such that every edge is incident to at least one vertex in S.

## **IP-Formulation of Vertex Cover**

$$\begin{array}{llll} \min & \sum_{v \in V} w_v x_v \\ \text{s.t.} & \forall e = (i,j) \in E & x_i + x_j & \geq & 1 \\ & \forall v \in V & x_v & \in & \{0,1\} \end{array}$$

# **Maximum Weighted Matching**

Given a graph G=(V,E), and a weight  $w_e$  for every edge  $e\in E$ . Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.

# **Maximum Independent Set**

Given a graph G=(V,E), and a weight  $w_v$  for every node  $v\in V$ . Find a subset  $S\subseteq V$  of nodes of maximum weight such that no two vertices in S are adjacent.

# Knapsack

Given a set of items  $\{1,\ldots,n\}$ , where the i-th item has weight  $w_i$  and profit  $p_i$ , and given a threshold K. Find a subset  $I\subseteq\{1,\ldots,n\}$  of items of total weight at most K such that the profit is maximized.

max		$\sum_{i=1}^{n} p_i x_i$		
s.t.		$\sum_{i=1}^n w_i x_i$	$\leq$	K
	$\forall i \in \{1, \dots, n\}$			{0,1}

### Relaxations

#### **Definition 4**

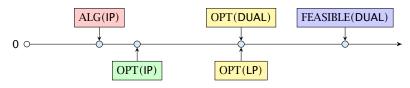
A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing  $x_i \in [0, 1]$  instead of  $x_i \in \{0, 1\}$ .

By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.

### Relations

#### **Maximization Problems:**



### **Minimization Problems:**

