Lemma 2 (Chernoff Bounds)

Let $X_1, \dots, X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$, $L \le \mu \le U$, and $\delta > 0$

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} ,$$

and

$$\Pr[X \le (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} ,$$

Lemma 3

For $0 \leq \delta \leq 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \leq e^{-L\delta^2/2}$$

Proof of Chernoff Bounds

Markovs Inequality:

Let X be random variable taking non-negative values.

Then

$$\Pr[X \geq a] \leq \mathrm{E}[X]/a$$

Trivial!

Proof of Chernoff Bounds

Hence:

$$\Pr[X \geq (1+\delta)U] \leq \frac{\mathrm{E}[X]}{(1+\delta)U} \approx \frac{1}{1+\delta}$$

That's awfully weak :(

Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

Cool Trick:

$$\Pr[X \geq (1+\delta)U] = \Pr[e^{tX} \geq e^{t(1+\delta)U}]$$

Now, we apply Markov:

$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} .$$

This may be a lot better (!?)

Proof of Chernoff Bounds

$$\mathrm{E}\left[e^{tX}\right] = \mathrm{E}\left[e^{t\sum_i X_i}\right] = \mathrm{E}\left[\prod_i e^{tX_i}\right] = \prod_i \mathrm{E}\left[e^{tX_i}\right]$$

$$\mathrm{E}\left[e^{tX_i}\right] = (1-p_i) + p_i e^t = 1 + p_i(e^t - 1) \leq e^{p_i(e^t-1)}$$

$$\prod_i \mathrm{E}\left[e^{tX_i}\right] \leq \prod_i e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)} = e^{(e^t-1)U}$$

Now, we apply Markov:

$$\begin{aligned}\Pr[X \geq (1+\delta)U] &= \Pr[e^{tX} \geq e^{t(1+\delta)U}] \\ &\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} \leq \frac{e^{(e^t-1)U}}{e^{t(1+\delta)U}} \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U\end{aligned}$$

We choose $t = \ln(1+\delta)$.

Lemma 4

For $0 \le \delta \le 1$ we have that

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \le e^{-L\delta^2/2}$$

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \leq -U\delta^2/3$$

True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \leq -2\delta/3$$

Reason:

As long as derivative of left side is smaller than derivative of right side the inequality holds.

$$f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$$

A convex function ($f''(\delta) \ge 0$) on an interval takes maximum at the boundaries.

$$f'(\delta) = -\frac{1}{1+\delta} + 2/3 \qquad f''(\delta) = \frac{1}{(1+\delta)^2}$$

$f(0) = 0$ and $f(1) = -\ln(2) + 2/3 < 0$

For $\delta \geq 1$ we show

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta/3}$$

Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \leq -U\delta/3$$

True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \leq -1/3 \iff \ln(1+\delta) \geq 1/3 \quad \text{(true)}$$

Reason:
As long as derivative of left side is smaller than derivative of right side the inequality holds.

Show:

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L \le e^{-L\delta^2/2}$$

Take logarithms:

$$L(-\delta - (1-\delta)\ln(1-\delta)) \le -L\delta^2/2$$

True for $\delta = 0$. Divide by L and take derivatives:

$$\ln(1-\delta) \le -\delta$$

Reason:

As long as derivative of left side is smaller than derivative of right side the inequality holds.

$$\ln(1-\delta) \leq -\delta$$

True for $\delta = 0$. Take derivatives:

$$-\frac{1}{1-\delta} \leq -1$$

This holds for $0 \leq \delta < 1$.

Integer Multicommodity Flows

- Given s_i-t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$\begin{array}{llrcl}
\min & & W & & \\
\text{s.t.} & \forall i & \sum_{p\in\mathcal{P}_i} x_p & = & 1 \\
 & & \sum_{p:e\in p} x_p & \le & W \\
 & & x_p & \in & \{0,1\}
\end{array}$$

Integer Multicommodity Flows

Randomized Rounding:
For each i choose one path from the set $\mathcal{P}_i$ at random according to the probability distribution given by the Linear Programming solution.

Theorem 5

If $W^* \geq c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

Theorem 6

With probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + c \ln n$.

Integer Multicommodity Flows

Let X_e^i be a random variable that indicates whether the path for s_i-t_i uses edge e.

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i : e \in p} x_p^* = \sum_{p : e \in P} x_p^* \leq W^*$$

Integer Multicommodity Flows

Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

19 MAXSAT

Problem definition:

- n Boolean variables
- m clauses $C_1, \dots, C_m$. For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- Non-negative weight w_j for each clause C_j.
- Find an assignment of true/false to the variables sucht that the total weight of clauses that are satisfied is maximum.

19 MAXSAT

Terminology:

- A variable x_i and its negation $\bar{x}_i$ are called literals.
- Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is **not** a clause).
- We assume a clause does not contain x_i and $\bar{x}_i$ for any i.
- x_i is called a positive literal while the negation $\bar{x}_i$ is called a negative literal.
- For a given clause C_j the number of its literals is called its length or size and denoted with ℓ_j.
- Clauses of length one are called unit clauses.

MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_j w_j X_j$$

$$
\begin{aligned}
E[W] &= \sum_j w_j E[X_j] \\
&= \sum_j w_j \Pr[C_j \text{ is satisified}] \\
&= \sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
&\geq \frac{1}{2} \sum_j w_j \\
&\geq \frac{1}{2} \mathrm{OPT}
\end{aligned}
$$

MAXSAT: LP formulation

- Let for a clause C_j, P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$\begin{array}{llrcl}
\max & & \sum_j w_j z_j & & \\
\text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\
& \forall i & y_i & \in & \{0,1\} \\
& \forall j & z_j & \leq & 1
\end{array}$$

MAXSAT: Randomized Rounding

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

Lemma 7 (Geometric Mean $\leq$ Arithmetic Mean)

For any nonnegative $a_1, \dots, a_k$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

Definition 8

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda)f(r)$$

Lemma 9

Let f be a concave function on the interval $[0,1]$, with $f(0) = a$ and $f(1) = a + b$. Then

$$\begin{aligned} f(\lambda) &= f((1-\lambda)0 + \lambda 1) \\ &\geq (1-\lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

for $\lambda \in [0,1]$.

$$
\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
&\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i)+\sum_{i\in N_j} y_i\right)\right]^{\ell_j} \\
&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j} \\
&\le \left(1-\frac{z_j}{\ell_j}\right)^{\ell_j} \quad .
\end{aligned}
$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^\ell$ is concave. Hence,

$$\begin{aligned}\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\ &\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \ .\end{aligned}$$

$f''(z) = -\frac{\ell-1}{\ell}\left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0,1]$. Therefore, f is concave.

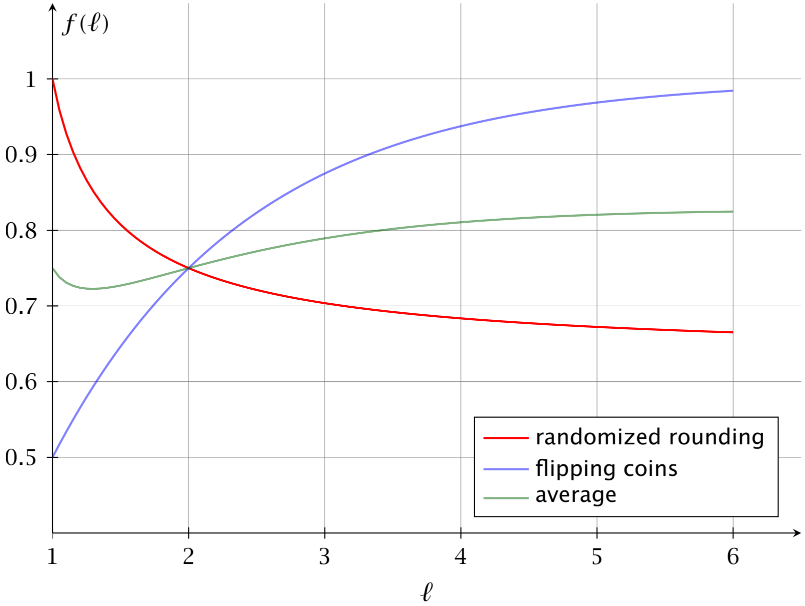
$$
\begin{aligned}
E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\
&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j} \right] \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \ .
\end{aligned}
$$

MAXSAT: The better of two

Theorem 10
Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$-approximation.

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$
\begin{aligned}
E[\max\{W_1, W_2\}]
&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
&\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{ for all integers}}\Bigg] \\
&\geq \frac{3}{4}\mathrm{OPT}
\end{aligned}
$$


$f(\ell)$
1
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6
ℓ
randomized rounding
flipping coins
average

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \rightarrow [0,1]$ and set x_i to true with probability $f(y_i)$.

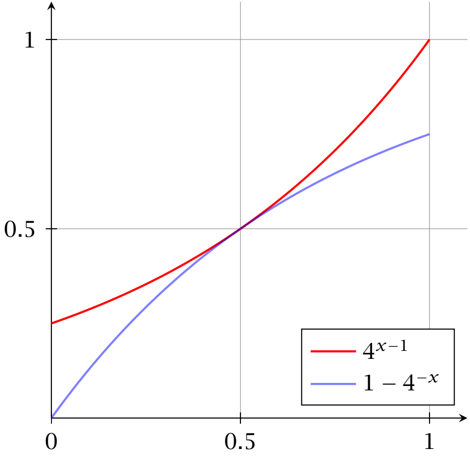
MAXSAT: Nonlinear Randomized Rounding

Let $f : [0,1] \rightarrow [0,1]$ be a function with

$$1 - 4^{-x} \leq f(x) \leq 4^{x-1}$$

Theorem 11

Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$-approximation.


1
0.5
0
0.5
1
4^{x-1}
$1 - 4^{-x}$

$$
\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-f(y_i))\prod_{i\in N_j} f(y_i)\\
&\le \prod_{i\in P_j} 4^{-y_i}\prod_{i\in N_j} 4^{y_i-1}\\
&= 4^{-(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i))}\\
&\le 4^{-z_j}
\end{aligned}
$$

The function $g(z) = 1 - 4^{-z}$ is concave on $[0, 1]$. Hence,

$$\Pr[C_j \text{ satisfied}] \geq 1 - 4^{-z_j} \geq \frac{3}{4} z_j \; .$$

Therefore,

$$E[W] = \sum_j w_j \Pr[C_j \text{ satisfied}] \geq \frac{3}{4} \sum_j w_j z_j \geq \frac{3}{4} \mathrm{OPT}$$

Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 12 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

Note that an integrality gap only holds for one specific ILP formulation.

Lemma 13

Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

$$
\begin{array}{llrcl}
\max & & \sum_j w_j z_j & & \\
\text{s.t.} & \forall j & \sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i) & \geq & z_j \\
& \forall i & y_i & \in & \{0,1\} \\
& \forall j & z_j & \leq & 1
\end{array}
$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- any solution can satisfy at most 3 clauses
- we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
- hence, the LP has value 4.