Deterministic Routing on the Multibutterfly Network

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Overview

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In contrast to the butterfly network

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- Robust against `faults` (i.e. congestion, failure).

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- `High-bandwidth` network.
- `Low-diameter` network.

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Definition 1

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3. for all $U \subseteq V$, $|U| \leq \alpha \cdot |V|$: $|\Gamma(u)| \geq \beta \cdot |U|$

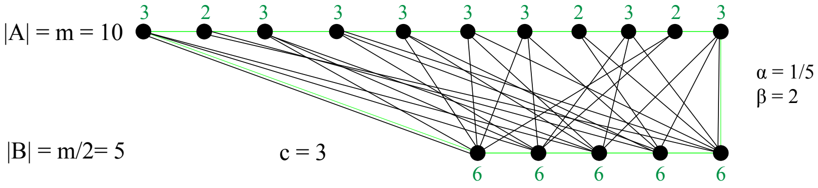

Figure: Design of a concentrator

Lemma 2: Existence of concentrators

For

$$\alpha \leq \frac{1}{2\beta}(4\beta \cdot e^{1+\beta})^{-\frac{1}{c-\beta-1}}$$

there exists an (α, β, m, c)*-concentrator.*

Proof of Lemma 2: Existence of concentrators

For an arbitrary permutation $\pi : A \rightarrow A$ *let*

$$E_\pi = \{(i, j) \in A \times B | \pi(i) \in \{j, j + \frac{m}{2}\}\}.$$

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Proof of Lemma 2: Existence of concentrators (continued)

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\end{aligned}$$

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\leq \quad & \prod_{i=1}^{c} \left(\frac{2 \cdot \beta \cdot \mu}{m} \right)^{\mu} = \left(\frac{2 \cdot \beta \cdot \mu}{m} \right)^{c \cdot \mu}
\end{aligned}
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$$\binom{k}{l} = \frac{k \cdot (k-1) \cdot \dots \cdot (k-l+1)}{l!} \leq \frac{k^l}{l!} = \frac{k^l}{l^l} \cdot \frac{l^l}{l!} \leq \frac{k^l}{l^l} \cdot e^l$$

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with $e^l = \sum_{i=0}^{\infty} \frac{l^i}{i!} \geq \frac{l^l}{l!}$

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$$\leq \sum_{\mu=1}^{\lfloor \alpha \cdot m \rfloor} \left[\left(\frac{m}{\mu} \right)^{1+\beta-c} \cdot e^{1+\beta} \cdot (2 \cdot \beta)^{c-\beta} \right]^{\mu}$$

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$$< \sum_{\mu=1}^{\infty} \left(\underbrace{\alpha^{c-1-\beta} \cdot e^{1+\beta} \cdot (2\beta)^{c-\beta}}_{=:r} \right)^{\mu} \qquad (\mu \leq \alpha \cdot m)$$

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Proof of Lemma 2: Existence of concentrators (continued)

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$$\alpha^{c-1-\beta} \cdot e^{1+\beta} \cdot (2\beta)^{c-\beta} \quad \leq \quad \frac{1}{2}$$

Proof of Lemma 2: Existence of concentrators (continued)
Necessary condition:

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&\dots \\
\alpha &\leq \frac{1}{2\beta} \cdot \left(4\beta \cdot e^{1+\beta}\right)^{-\frac{1}{c-1-\beta}}
\end{aligned}$$

□

Lemma 2: Existence of concentrators
For $\alpha \leq \frac{1}{2\beta}(4\beta \cdot e^{1+\beta})^{-\frac{1}{c-\beta-1}}$ *there exists a* (α, β, m, c)*-concentrator.*

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Example
For $\beta = 2$ *and* $c = 4$ *and* $\alpha \leq \frac{1}{32e^3}$ *there exists a* $(\frac{1}{643}, 2, m, 4)$*-concentrator.*

Overview

Definition 3

A (α, β, m, c)`-splitter` *is a bipartite graph* $G = (V \cup (W_0 \cup W_1), E_0 \cup E_1)$ *in which* $(V \cup W_0, E_0)$ *and* $(V \cup W_1, E_1)$ *represent* (α, β, m, c)*-concentrators.*

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Edges in E_0 *are called 0-edges, and edges in* E_1 *are called 1-edges.*

Splitter

Design of a splitter

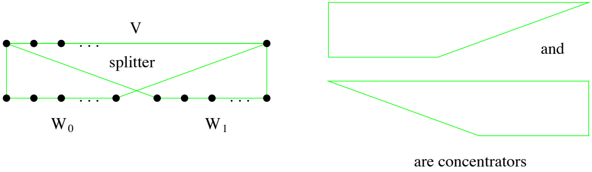

Figure: Design of a splitter

Overview

Definition 4

The `d-dimensional multibutterfly graph` (d, α, β, c) *has*

$$N = \underbrace{(d+1)}_{number\ of\ levels} \cdot \underbrace{2^d}_{number\ of\ inputs/outputs}$$

nodes and degree at most $4c$.

The Multibutterfly Graph

Recursive construction

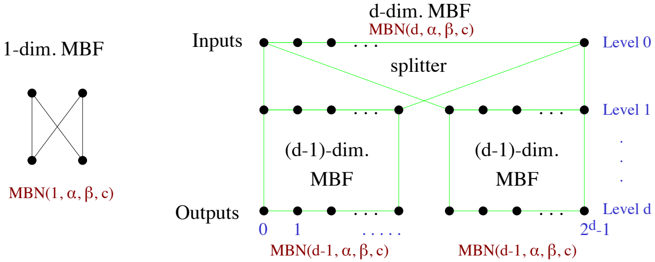

Figure: Recursive construction of a multibutterfly graph

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  - $j \mod L = i$
- The waves $A_0, \ldots, A_{L-1}$ consist of approx. $\frac{n}{L}$ packets.

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  - `odd` phase (sending from odd to even levels)
    - each phase consisting of `2c steps`
- The edges connecting levels are `colored` in 2c colors
  - so that each node is incident to one edge of each color (`matching`)

Routing on the MBN

Unique logical path

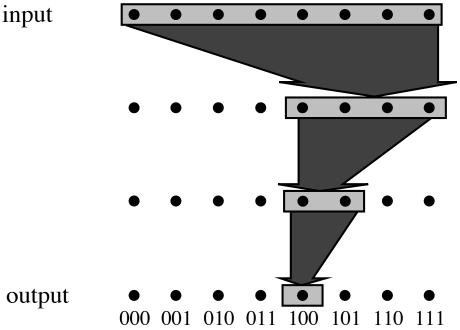

Figure: The unique logical path between an input and an output.

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The algorithm

- Input: packets p_0, ..., p_n with destinations j ($n = 2^d$)

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- Partitioning the packets into waves $A_0, \ldots, A_{L-1}$ by destination j so that j mod L = i (for A_i)

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      . For j := 1 to $2c$ do // colors of the edges

Routing on the MBN

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- Foreach wave $A_i \in A_0, \ldots, A_{L-1}$ do // the waves
  - While $\exists p_k \in A_i$ do // packets not by the output
    - Foreach node $n_{e/o} \in \{even\ \ level\}$, $n_{o/e} \in \{odd\ \ level\}$

      . For j := 1 to $2c$ do // colors of the edges

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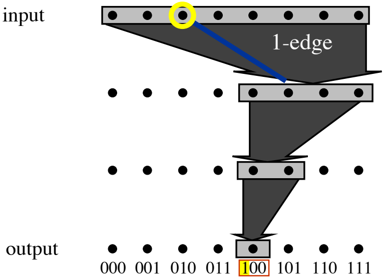
Routing on the MBN

The algorithm - example



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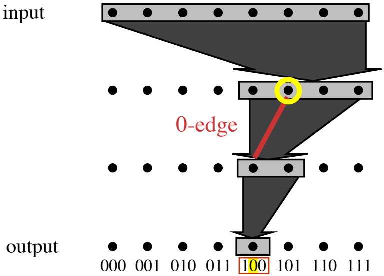

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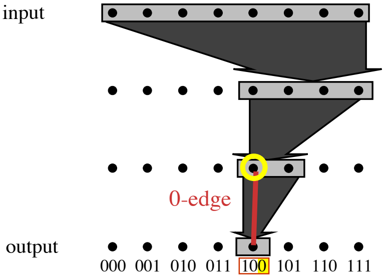

Overview

The analysis

The phases (even and odd levels):

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How many phases are necessary for one wave?

- The number of phases until all packets of one wave are at the outputs.

Approach

- Suppose `t ≥ 0` the `number of phases`.

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The analysis

The number of phases in one wave

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- Let K_i be the `number of packets` after t phases in `level` i …
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- Assume b_i to be the number of `blocked packets` after $t+1$ phases in level i …
- …and a_i the number of `active packets` (recently sent) after $t+1$ phases in level i.
- $\rightarrow \quad K_i' = a_i + b_i$

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d) $b_{d-1} = 0$: No packets are blocked in the level second to last.

Proof of $b_i \leq \frac{1}{\beta+1} \cdot (K_{i+1} + K_i)$ for i $\leq$ d - 2

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- $$\frac{1}{2} \cdot \frac{m}{L} \approx \frac{1}{2} \cdot \frac{m}{\frac{1}{2\alpha}} = \alpha \cdot m$$

  packets take the 0- or 1-edge in the (α, β, m, c)-splitter.

Proof of $b_i \leq \frac{1}{\beta+1} \cdot (K_{i+1} + K_i)$ for i $\leq$ d - 2

- Let z be the number of blocked packets in round $t + 1$, so:

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- Expansion-property of the (α, β, m, c)-concentrator: $\alpha \cdot z$ are blocked by at least $\beta \cdot z$ packets.
- Round $t+1$: there are at most $K_{i+1} + a_{i+1}$ blocking packets on level $i+1$:

Proof of $b_i \leq \frac{1}{\beta+1} \cdot (K_{i+1} + K_i)$ for i $\leq$ d - 2

$$\beta \cdot b_i \leq K_{i+1} + a_{i+1} =^{b)} K_{i+1} + K_i - b_i$$
$$\Rightarrow b_i \cdot (1+\beta) \leq K_{i+1} + K_i$$
$$\Rightarrow b_i \leq \frac{1}{1+\beta} \cdot (K_{i+1} + K_i)$$

□

The analysis
The running-time

We will analyze the running time by means of a `potential function argument`:

- Suppose $\omega \in (0, 1)$.

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$\rightarrow$ Packets are `weighted` depending on their distance to the outputs.

Lemma 6

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c) $\phi_{t+1} \leq \phi_t \cdot \left(\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\cdot\omega}\right)$.

Proof of Lemma 6 a): $\phi_0 = \frac{n}{L}$

- At the beginning there are $\frac{n}{L}$ packets in Level 0, i.e. $K_0 = \frac{n}{L}, K_i \; = \; 0$ for $i = 1, \ldots, d-1$.

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Proof of Lemma 6 b): $\phi_t < \omega^{d-1}$

- If $\phi_t < \omega^{d-1}$ then $\phi_t = 0$ and all $K_i = 0$, i.e. all packets are on level d at the outputs.

Proof of Lemma 6 c): $\phi_{t+1} \leq \phi_t \cdot \left(\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\cdot\omega}\right)$

The analysis

The running-time

Proof of Lemma 6 c): $\phi_{t+1} \leq \phi_t \cdot \left(\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\cdot\omega}\right)$

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&\leq \sum_{i=1}^{d-1} K_{i-1}\omega^i + \sum_{i=0}^{d-2} \frac{1}{\beta+1}(K_{i+1} + K_i)(\omega^i - \omega^{i+1})
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$$= \phi_t \cdot \left(\left(1 - \frac{1}{\beta+1}\right)\omega + \frac{1}{(\beta+1)\omega}\right)$$

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$\Longrightarrow$ *If* $\left(\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\omega}\right) < 1$, ϕ_t *converges to* 0

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Theorem 7
Suppose an arbitrary $\beta > 1$, let α and c be chosen, so that there exists a MBN(d, α, β, c) for any d.
So, the MBN(d, α, β, c) can route arbitrary permutations of $n = 2^d$ packets in

$$O(\log n).$$

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Proof of Theorem 7

Assume $\delta := \frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\omega}$

Now we have to choose $\omega \in (0, 1)$ *and* $\delta \in (0, 1)$

Proof of Theorem 7 (continued)

$$\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\omega} = \delta$$

Proof of Theorem 7 (continued)

$$
\begin{aligned}
\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\omega} &= \delta \\
\Leftrightarrow \omega^2 - \delta \cdot \frac{\beta+1}{\beta} \cdot \omega + \frac{1}{\beta} &= 0
\end{aligned}
$$

Proof of Theorem 7 (continued)

$$\begin{aligned}
\frac{\beta}{\beta+1}\omega + \frac{1}{(\beta+1)\omega} &= \delta \\
\Leftrightarrow \omega^2 - \delta \cdot \frac{\beta+1}{\beta} \cdot \omega + \frac{1}{\beta} &= 0 \\
\Leftrightarrow \omega_{1/2} &= \frac{1}{2}\frac{\delta(\beta+1)}{\beta} \pm \sqrt{\left(\frac{\delta(\beta+1)}{2\beta}\right)^2 - \frac{1}{\beta}}
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Proof of Theorem 7 (continued)

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Choose $\delta = \frac{2\sqrt{\beta}}{\beta+1} \Longrightarrow \sqrt{\ldots} = 0 \Longrightarrow$ Unique solution.

Proof of Theorem 7 (continued)

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So, $\omega = \frac{1}{2}\frac{\beta+1}{\beta}\left(\frac{2\sqrt{\beta}}{\beta+1}\right) = \frac{1}{\sqrt{\beta}}$.

Proof of Theorem 7 (continued)

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Choose $\delta = \frac{2\sqrt{\beta}}{\beta+1} \Longrightarrow \sqrt{\ldots} = 0 \Longrightarrow$ Unique solution.

So, $\omega = \frac{1}{2}\frac{\beta+1}{\beta}\left(\frac{2\sqrt{\beta}}{\beta+1}\right) = \frac{1}{\sqrt{\beta}}$.
With $\beta > 1$, ω will be less than 1.

Proof of Theorem 7 (continued)

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- In every phase, ϕ_T decreases, so $\phi_T \leq \frac{n}{L} \cdot \delta^T$.
- For the first T with $\phi_T < \omega^{d-1}$: T phases are `sufficient` for one wave. (Lemma 6 b))

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We have to specify $min\{T \mid \phi_T < \omega^{d-1}\}$.

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$$\frac{n}{L} \cdot \delta^T \quad < \quad \omega^{d-1}$$

Proof of Theorem 7 (continued)

We have to specify $min\{T \mid \phi_T < \omega^{d-1}\}$.

$$\begin{aligned} \frac{n}{L} \cdot \delta^T &< \omega^{d-1} \\ \Leftrightarrow \delta^T &< \omega^{d-1} \cdot \frac{L}{n} \end{aligned}$$

Proof of Theorem 7 (continued)

We have to specify $min\{T \mid \phi_T < \omega^{d-1}\}$.

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□

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