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- An exact analysis (e.g. exactly counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.
- A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.
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Formal Definition

Let f denote functions from \mathbb{N} to \mathbb{R}^+ .

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \ \exists n_0 \in \mathbb{N}_0 \ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$ (set of functions that asymptotically grow not faster than f)



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There is an equivalent definition using limes notation. f and g are functions from $\mathbb N$ to $\mathbb R^+$.

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- 1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).
- 2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f: \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g: \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.
- 3. People write e.g. h(n) = f(n) + o(g(n)) when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) \leq f(n) + z(n)$.
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Lemma 3

Let f, g be functions with the property $\exists n_0 > 0 \ \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- $c \cdot f(n) \in \Theta(f(n))$ for any constant c
- $\bullet \ \mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$
- $\triangleright \mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(\max\{f(n), g(n)\})$



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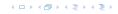
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Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.



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