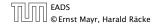
First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\left\lceil \frac{n}{2} \right\rceil) + cn & n \ge 2\\ 0 & \text{otherwise} \end{cases}$$

Assume that instead we had

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 2\\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.



Suppose we guess  $T(n) \le dn \log n$  for a constant *d*. Then

$$T(n) \le 2T\left(\frac{n}{2}\right) + cn$$
  
$$\le 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn$$
  
$$= dn(\log n - 1) + cn$$
  
$$= dn\log n + (c - d)n$$
  
$$\le dn\log n$$

if we choose  $d \ge c$ .

Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

**Guess:**  $T(n) \le dn \log n$ . **Proof.** (by induction)

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 16\\ b & \text{otw.} \end{cases}$$

- **base case**  $(2 \le n < 16)$ : true if we choose  $d \ge b$ .
- induction step  $2 \dots n 1 \rightarrow n$ :

Suppose statem. is true for  $n' \in \{2, ..., n-1\}$ , and  $n \ge 16$ . We prove it for n:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn$$

$$= dn(\log n - 1) + cn$$

$$= dn\log n + (c - d)n$$
• Note that this proves the statement for  $n \in \mathbb{N}_{\geq 2}$ , as the statement is wrong for  $n = 1$ .
• The base case is usually omitted, as it is the same for different recurrences.

 $\leq dn \log n$ 

Hence, statement is true if we choose  $d \ge c$ .

Why did we change the recurrence by getting rid of the ceiling?

If we do not do this we instead consider the following recurrence:

$$T(n) \le \begin{cases} 2T(\left\lceil \frac{n}{2} \right\rceil) + cn & n \ge 16\\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant (*b* in the above case).



We also make a guess of  $T(n) \le dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1 \leq 2\left(d(n/2 + 1)\log(n/2 + 1)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4) = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4} \leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d.

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