# Technique 2: Rounding the Dual Solution.

**Relaxation for Set Cover**

**Primal:**

$$\begin{array}{lrl} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

**Dual:**

$$\begin{array}{lrl} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

# Technique 2: Rounding the Dual Solution.

**Relaxation for Set Cover**

**Primal:**

$$\begin{array}{lrl} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

**Dual:**

$$\begin{array}{lrl} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

# Technique 2: Rounding the Dual Solution.

**Relaxation for Set Cover**

**Primal:**

$$\begin{array}{lrl} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

**Dual:**

$$\begin{array}{lrl} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

# Technique 2: Rounding the Dual Solution.

**Rounding Algorithm:**

Let $I$ denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u\in S_i} y_u = w_i$$

# Technique 2: Rounding the Dual Solution.

**Lemma 3**

*The resulting index set is an $f$-approximation.*

**Proof:**

Every $u \in U$ is covered.

- Suppose there is a $u$ that is not covered.
- This means $\sum_{u:u\in S_j} y_u < w_j$ for all sets $S_j$ that contain $u$.
- But then $y_u$ could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

# Technique 2: Rounding the Dual Solution.

**Lemma 3**

*The resulting index set is an $f$-approximation.*

**Proof:**

Every $u \in U$ is covered.

- Suppose there is a $u$ that is not covered.
- This means $\sum_{u: u \in S_j} y_u < w_j$ for all sets $S_j$ that contain $u$.
- But then $y_u$ could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

# Technique 2: Rounding the Dual Solution.

**Lemma 3**

*The resulting index set is an $f$-approximation.*

**Proof:**

Every $u \in U$ is covered.

- Suppose there is a $u$ that is not covered.
- This means $\sum_{u:u\in S_i} y_u < w_i$ for all sets $S_i$ that contain $u$.
- But then $y_u$ could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

# Technique 2: Rounding the Dual Solution.

**Lemma 3**

*The resulting index set is an $f$-approximation.*

**Proof:**

Every $u \in U$ is covered.

- Suppose there is a $u$ that is not covered.
- This means $\sum_{u:u\in S_i} y_u < w_i$ for all sets $S_i$ that contain $u$.
- But then $y_u$ could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

# Technique 2: Rounding the Dual Solution.

**Lemma 3**

*The resulting index set is an $f$-approximation.*

**Proof:**

Every $u \in U$ is covered.

- Suppose there is a $u$ that is not covered.
- This means $\sum_{u:u\in S_i} y_u < w_i$ for all sets $S_i$ that contain $u$.
- But then $y_u$ could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\sum_{i\in I} w_i$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\sum_{i\in I} w_i = \sum_{i\in I}\sum_{u:u\in S_i} y_u$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\begin{aligned}\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\ &= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u\end{aligned}$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\sum_{i\in I} w_i = \sum_{i\in I}\sum_{u:u\in S_i} y_u$$

$$= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u$$

$$\le \sum_u f_u y_u$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$
\begin{aligned}
\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\
&= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u \\
&\le \sum_u f_u y_u \\
&\le f\sum_u y_u
\end{aligned}
$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\begin{aligned}\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\ &= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u \\ &\le \sum_u f_u y_u \\ &\le f\sum_u y_u \\ &\le f\,\mathrm{cost}(x^*)\end{aligned}$$

# Technique 2: Rounding the Dual Solution.

**Proof:**

$$\begin{aligned}
\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\
&= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u \\
&\le \sum_u f_u y_u \\
&\le f\sum_u y_u \\
&\le f\operatorname{cost}(x^*) \\
&\le f\cdot \mathrm{OPT}
\end{aligned}$$

Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i^* \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.

Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.

Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.

Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.

Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.